Trabalho de Conclusão de Curso - Bacharelado em Matemática

Structure and Classification of Clifford Algebras

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"Tentaram me enterrar vivo, mas eu estou aqui".

RESUMO

O objetivo principal desse trabalho é investigar a estrutura das álgebras de Clifford e desenvolver a sua classificação. Apresentamos um estudo detalhado das álgebras de Clifford, sua definição e construção explícita como quociente da álgebra tensorial, suas propriedades gerais, os principais teoremas de sua estrutura e a construção da classificação dessas álgebras baseada na sua periodicidade por meio da teoria de representação.

Palavras Chaves: álgebra, álgebras de Clifford, classificação

ABSTRACT

The main goal of this work is to investigate the structure of Clifford algebras and develop its classification. We present a detailed study of the Clifford algebras, the definition and its explicit construction as quotient of tensor algebra, the general properties, the main theorems on its structure and the construction of the classification of those algebras based on their periodicity through representation theory.

Keywords: algebra, Clifford algebras, classification

INTRODUCTION

In the beginning of the 20th century there was a great interest in developing a theory that describes the electron. Due to the electron's nature, this theory must have a quantum and a relativistic approach. The Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\psi + W\psi, \tag{0.1}$$

describes all atomic phenomena except those involving magnetism and relativity. Due to the distinct orders of time and space derivatives on that equation the Schrödinger equation is not suitable to deal with relativistic phenomena. The different orders of derivatives do not make the equation invariant under space-time transformations and this contradicts the principle that the laws of physics must be the same for all inertial observers. Therefore, since the Schrödinger equation is incompatible with the theory of relativity another path was to start from the relativistic energy equation,

$$E^2 = \mathbf{p}^2c^2 + m^2c^2 \tag{0.2}$$

By the correspondence principle, inserting energy and momentum operators

$$E = i\hbar\frac{\partial}{\partial t}, \qquad \mathbf{p} = -i\hbar\nabla, \tag{0.3}$$

into that equation, results in the Klein-Gordon equation

$$\hbar^2\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\psi = m^2c^2\psi. \tag{0.4}$$

The Klein-Gordon equation solves the asymmetry problem in the derivatives, however, only for spinless particles. Also, this equation has a second derivative problem that can give rise to what would be a probability density that is not always positive. Therefore, what the physicist Paul Dirac did was linearise the Klein-Gordon equation, or replaced it with a first-order equation,

$$i\hbar\left(\gamma_0\frac{1}{c^2}\frac{\partial}{\partial t} + \gamma_1\frac{\partial}{\partial x_1} + \gamma_2\frac{\partial}{\partial x_2} + \gamma_3\frac{\partial}{\partial x_3}\right)\psi = mc\psi, \tag{0.5}$$

this way the derivative in time would become linear, therefore time and space are treated on an equal footing which solves the problem that the Klein-Gordon equation was facing. In addition, Dirac brought very important interpretations such as the existence of antiparticles as an explanation for the existence of negative energies that the Klein-Gordon equation presents. Hence, Dirac's theory takes into account the invariance under space-time isometries and also takes into account spins' interaction, in particular, the Dirac equation considers relativistic effects as well as spin-$\frac{1}{2}$ particles achieving the main goal of establishing a theory that describes the electron [1]. However, the development of this theory was only possible because Dirac introduced a set of interesting 4×4 complex matrices γ_μ, $\mu = 0, 1, 2, 3$:

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ \gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{0.6}$$

satisfying a very specific relation, that is:

$$\begin{aligned} &\gamma_0^2 = I, \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -I, \\ &\gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu \quad \text{for} \quad \mu \neq \nu. \end{aligned} \tag{0.7}$$

or in terms of the metric $\eta_{\mu\nu}$ of the Minkowski space-time $\mathbb{R}^{1,3}$

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I \tag{0.8}$$

This seemingly innocent condition is in fact quite deep. It first appeared 1878, in an attempt to unify the structures introduced by Grassmann and Hamilton, namely, the exterior algebra and the quaternions respectively. William Kingdom Clifford in his work *"Applications Of Grassmann's extensive algebra"*, introduced the analogue of the quaternion product on the multivector structure of exterior algebra in which he established a new algebraic structure called Geometric Algebra or Clifford algebra. In contrast to the quaternion algebra or the Gibbs algebra, this new structure is not limited to be defined on a particular space, but in any quadratic space, dimensions and signatures, which makes Clifford algebras rich in structure and its applications.

Our purpose in this work is the detailed study of Clifford algebras in general, that is, in any dimension and signature, presenting their formal definition, the properties of its structure and their classification.

The first Chapter provides an introduction to the theory of Clifford algebras: the definition, the explicit construction as a quotient of the tensor algebra by a two-sided ideal, we will show that in any quadratic space is possible to define a Clifford algebra and we will present some examples. The second Chapter provides a detailed exposition of the Clifford algebra structure. We will present some operations that can be defined on those algebras, some important subspaces and as the main goal of this Chapter: the main theorems concerning the Clifford algebras structure that unravel some important behaviour of these algebras like its periodicity, that allow us in the next Chapter, present the objective of this work: the Clifford algebras classification. Finally, in the third Chapter we will explore the periodicity of the Clifford algebras and the classification will be implemented.

1 DEFINITION OF CLIFFORD ALGEBRAS

This Chapter provides an introduction to the theory of the Clifford algebras: the definition, the explicit construction as a quotient of the tensor algebra and some examples.

1.1. Definition

To define a Clifford (real) algebra, a vector space over $\mathbb{R}$ and a quadratic form Q defined in V are the basic ingredients. A Clifford algebra is an algebra generated from the vectors of V in such a way that the square of an element of V is related to the quadratic form. In order to establish this algebraic structure let us characterise an algebra first.

Definition 1.1. *An* ***algebra*** *over a filed* $\mathbb{K}$ *is a vector space* $\mathcal{A}$ *over* $\mathbb{K}$ *endowed with a bilinear product* $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *,* $(a,b) \to ab$*.*

In other words, let V be a $\mathbb{K}$-vector space, the pair $\mathcal{A} = (V,*)$ such that $* : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called an algebra over $\mathbb{K}$ if given $a,b,c \in V$ and $\lambda \in \mathbb{K}$ the product $*$ satisfies the following conditions [8]:

(i) $a*(b+c) = a*b = a*c,$

(ii) $(a+b)*c = a*c+b*c,$

(iii) $\lambda(a*b) = (\lambda a)*b = a*(\lambda b).$

These conditions are the bilinearity of $*$. Furthermore, if there exists an element $e \in V$ such that $e*a = a*e = a, \ \forall a \in V$, we say that we have an algebra with unity. Moreover, if $a*b = b*a, \ \forall a,b \in V$ we have a commutative algebra and if $(a*b)*c = a*(b*c), \ \forall a,b,c \in V$ we have an associative algebra.

Definition 1.2. *Let* V *be a vector space over* $\mathbb{R}$ *equipped with a symmetric bilinear form* g*. Let* $\mathcal{A}$ *be an associative algebra with unity* $1_{\mathcal{A}}$ *and let* γ *be the linear mapping* $\gamma : V \to \mathcal{A}$*. The pair* $(\mathcal{A},\gamma)$ *is a* ***Clifford algebra*** *for the quadratic space* (V,g) *when* $\mathcal{A}$ *is* *generated as an algebra by* $\{\gamma(\mathbf{v}) \mid \mathbf{v} \in V\}$ *and* $\{x1_{\mathcal{A}} \mid x \in \mathbb{R}\}$*, and* γ *satisfies for all* $\mathbf{v}, \mathbf{u} \in V$ *the relation*

$$\gamma(\mathbf{v})\gamma(\mathbf{u}) + \gamma(\mathbf{u})\gamma(\mathbf{v}) = 2g(\mathbf{u},\mathbf{v})1_{\mathcal{A}}. \tag{1.1}$$

The mapping γ is called *Clifford mapping* and is a kind of square root of the quadratic form $Q(\mathbf{v}) = g(\mathbf{v},\mathbf{v})$ since Eq. (1.1) can be written as

$$\gamma(\mathbf{v})^2 = Q(\mathbf{v})1_{\mathcal{A}} = g(\mathbf{v},\mathbf{v})1_{\mathcal{A}}, \tag{1.2}$$

for all $\mathbf{v} \in V$.

Some properties of Clifford algebras can be explored noticing how the relation (1.1) works with respect to an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V. For a Clifford algebra $(\mathcal{A}, \gamma)$ for (V, g) we have

$$\begin{aligned} \gamma(\mathbf{e}_i)\gamma(\mathbf{e}_j) + \gamma(\mathbf{e}_j)\gamma(\mathbf{e}_i) = 0, \quad \text{for } i \neq j. \\ \gamma(\mathbf{e}_i)^2 = g(\mathbf{e}_i,\mathbf{e}_i)1_{\mathcal{A}}. \end{aligned} \tag{1.3}$$

By using those relations, any product involving $\gamma(\mathbf{e}_i)$ and their powers can be reordered to yield [7]

$$\gamma(\mathbf{e}_1)^{\mu_1}\gamma(\mathbf{e}_2)^{\mu_2}\ldots\gamma(\mathbf{e}_n)^{\mu_n}, \quad \mu_i = 0,1, \ (i = 1,\ldots,n) \tag{1.4}$$

and since $\mathcal{A}$ is generated by $\{\gamma(\mathbf{v}) \mid \mathbf{v} \in V\}$ and $\{x1_{\mathcal{A}} \mid x \in \mathbb{R}\}$, it follows that it is generated by the products

$$\mathcal{A} = \text{span}\{\gamma(\mathbf{e}_1)^{\mu_1}\gamma(\mathbf{e}_2)^{\mu_2}\ldots\gamma(\mathbf{e}_n)^{\mu_n} \ \mid \ \mu_i = 0,1\}, \tag{1.5}$$

such that the identity is denoted as $\gamma(\mathbf{e}_1)^0\gamma(\mathbf{e}_2)^0\ldots\gamma(\mathbf{e}_n)^0 = 1_{\mathcal{A}}$. One can notice that the maximum number of elements of type $\gamma(\mathbf{e}_1)^{\mu_1}\gamma(\mathbf{e}_2)^{\mu_2}\ldots\gamma(\mathbf{e}_n)^{\mu_n}$ with $\mu_i = 0,1$ is 2^n, then, the maximal dimension of a Clifford algebra is 2^n. Such algebras are said to be a *universal Clifford algebra* [7].

Definition 1.3. *A Clifford algebra* $(\mathcal{A}, \gamma)$ *for the quadratic space* (V, g) *is said to be an* ***universal Clifford algebra*** *if for each Clifford algebra* $(\mathcal{B}, \rho)$ *for* (V, g) *there exists a unique homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$ *such that* $\rho = \phi \circ \gamma$*. In these conditions, a universal Clifford algebra for* (V, g) *is denoted by* $\mathcal{C}\ell(V, g)$*.*

The universal Clifford algebra $\mathcal{C}\ell(V, g)$, if it exists, it is unique up to a unique homomorphism. In other words, the above definition state that for each linear application $\rho : V \to \mathcal{B}$ there is a unique algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ making the following diagram commute

$$\begin{tikzcd} V \arrow[r, "\gamma"] \arrow[d, "\rho"'] & \mathcal{A} \arrow[dl, dashed, "\phi"] \\ \mathcal{B} & \end{tikzcd}$$

Thereby, we know that if a universal Clifford algebra $\mathcal{C}\ell(V,g)$ exists, by the uniqueness of the homomorphism, $\mathcal{C}\ell(V,g)$ will be unique. One can ask if for any quadratic space (V,g) given, there exists a universal Clifford algebra $\mathcal{C}\ell(V,g)$ associated to this space. The answer is affirmative and we will show its explicit construction as the quotient of the tensor algebra of V by a specific two-sided ideal, in a similar way as it is done for the exterior algebra in the Appendix C. Before, we will present a result that concerns the dimension of the universal Clifford algebras.

Theorem 1.4. *Let* $(\mathcal{A},\gamma)$ *be a Clifford algebra for the quadratic space* (V,g)*. If dim* $\mathcal{A} = 2^n$ *with* $n = \dim V$ *then* $(\mathcal{A},\gamma)$ *is a universal Clifford algebra for* (V,g)*.*

Proof. Consider $\{\mathbf{e}_1,...,\mathbf{e}_n\}$, an orthonormal basis of V. Since $(\mathcal{A},\gamma)$ is a Clifford algebra the relation (1.3) holds. Moreover, from the hypothesis we have that [dim $\mathcal{A} = 2^n$] therefore the set $\{\gamma(\mathbf{e}_1)^{\mu_1}\gamma(\mathbf{e}_2)^{\mu_2}...\gamma(\mathbf{e}_n)^{\mu_n} \mid \mu_i = 0,1\}$ generates and is a basis for $\mathcal{A}$. Now, let $(\mathcal{B},\rho)$ be an arbitrary Clifford algebra for (V,g). It holds the same relation (1.3) for $\mathcal{B}$ thence the set $\{\rho(\mathbf{e}_1)^{\mu_1}\rho(\mathbf{e}_2)^{\mu_2}...\rho(\mathbf{e}_n)^{\mu_n} \mid \mu_i = 0,1\}$ generates $\mathcal{B}$. Right, define the linear mapping $\phi : \mathcal{A} \to \mathcal{B}$ such that

$$\phi(\gamma(\mathbf{e}_1)^{\mu_1}\gamma(\mathbf{e}_2)^{\mu_2}...\gamma(\mathbf{e}_n)^{\mu_n}) = \rho(\mathbf{e}_1)^{\mu_1}\rho(\mathbf{e}_2)^{\mu_2}...\rho(\mathbf{e}_n)^{\mu_n}. \tag{1.6}$$

That way defined, ϕ is an algebra isomorphism satisfying

$$\phi(\gamma(\mathbf{e}_j))\phi(\gamma(\mathbf{e}_i)) + \phi(\gamma(\mathbf{e}_i))\phi(\gamma(\mathbf{e}_j)) = 2g(\mathbf{e}_i,\mathbf{e}_j)1_{\mathcal{B}}. \tag{1.7}$$

Thus, by the Definition 1.3, $(\mathcal{A},\gamma)$ is a universal Clifford algebra $\mathcal{C}\ell(V,g)$. □

1.2. Clifford Algebra as Quotient of Tensor Algebra

Theorem 1.5. *For all quadratic space* (V,g) *there exists a universal Clifford algebra.*

Proof. Let (V,g) be a quadratic space and $T(V)$ the algebra of the contravariant tensors. In order to construct a Clifford algebra as a quotient of the tensor algebra by a two-sided ideal, is required that the elements of our Clifford algebra obey the fundamental relation (1.2), let us consider then the ideal $\mathcal{I}$ of $T(V)$ generated by elements of type

$$\mathbf{v}\otimes\mathbf{v}-Q(\mathbf{v})1_T \tag{1.8}$$

where $Q(\mathbf{v})=g(\mathbf{v},\mathbf{v})$ and $1_{T(V)}$ is the identity of the tensor algebra. That way, the two-sided ideal $\mathcal{I}$ consists of all sums

$$\sum_i A_i\otimes(\mathbf{v}\otimes\mathbf{v}-Q(\mathbf{v})1_T)\otimes B_i \tag{1.9}$$

where $A_i,B_i\in T(V)$. However, one can also realise that the ideal $\mathcal{I}$ is generated equivalently by the elements

$$\mathbf{v}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{v}-2g(\mathbf{v},\mathbf{u})1_T. \tag{1.10}$$

Consider now the quotient map $\pi: T(V)\to T(V)/\mathcal{I}$ and the inclusion $\iota: V\hookrightarrow T(V)$ then one can define $\gamma: V\to T(V)/\mathcal{I}$ such that the diagram below commutes.

$$\begin{tikzcd}
V \arrow[r, "\iota"] \arrow[dr, "\gamma"'] & T(V) \arrow[d, "\pi"] \\
 & T(V)/\mathcal{I}
\end{tikzcd}$$

Now let us prove that $\gamma=\pi\circ\iota$ is a Clifford mapping. Consider the equivalence relation

$$A\sim B \iff A=B+x,\quad x\in\mathcal{I}. \tag{1.11}$$

Let $\mathbf{u},\mathbf{v}\in V$ and notice that

$$\gamma(\mathbf{v})\gamma(\mathbf{u})=\pi(\iota(\mathbf{v})\pi(\iota(\mathbf{u}))=\pi(\iota(\mathbf{v})\otimes(\iota(\mathbf{u}))=[\mathbf{v}\otimes\mathbf{u}]. \tag{1.12}$$

Such that $[\mathbf{v}\otimes\mathbf{u}]$ is the equivalence class of $\mathbf{v}\otimes\mathbf{u}$. In addition, we have that $\mathbf{v}\otimes\mathbf{u}$ can be expressed as

$$\begin{aligned}
\mathbf{v}\otimes\mathbf{u} &= \frac{1}{2}(\mathbf{v}\otimes\mathbf{u}-\mathbf{u}\otimes\mathbf{v})+g(\mathbf{v},\mathbf{u})1_T+\frac{1}{2}(\mathbf{v}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{v})-g(\mathbf{v},\mathbf{u})1_T\\
&= \frac{1}{2}(\mathbf{v}\otimes\mathbf{u}-\mathbf{u}\otimes\mathbf{v})+g(\mathbf{v},\mathbf{u})1_T+\frac{1}{2}[(\mathbf{v}+\mathbf{u})\otimes(\mathbf{v}+\mathbf{u})-g(\mathbf{v}+\mathbf{v},\mathbf{v}+\mathbf{u})1_T\\
&\quad -\mathbf{v}\otimes\mathbf{v}+g(\mathbf{v},\mathbf{v})1_T-\mathbf{u}\otimes\mathbf{u}+g(\mathbf{u},\mathbf{u})1_T].
\end{aligned} \tag{1.13}$$

As we can see, the term in the brackets is an element of the ideal $\mathcal{I}$, therefore it yields

$$\mathbf{v}\otimes\mathbf{u}\sim\frac{1}{2}(\mathbf{v}\otimes\mathbf{u}-\mathbf{u}\otimes\mathbf{v})+g(\mathbf{v},\mathbf{u})1_T \tag{1.14}$$

or equivalently

$$\mathbf{v}\otimes\mathbf{u}\sim\mathbf{v}\wedge\mathbf{u}+g(\mathbf{v},\mathbf{u})1_T. \tag{1.15}$$

Hence,

$$\begin{aligned}\gamma(\mathbf{v})\gamma(\mathbf{u})+\gamma(\mathbf{u})\gamma(\mathbf{v})&=[\mathbf{v}\otimes\mathbf{u}]+[\mathbf{u}\otimes\mathbf{v}]\\&=[\mathbf{v}\wedge\mathbf{u}+g(\mathbf{v},\mathbf{u})1_T]+[\mathbf{u}\wedge\mathbf{v}+g(\mathbf{u},\mathbf{v})1_T]\\&=2g(\mathbf{v},\mathbf{u})1_T,\end{aligned} \tag{1.16}$$

which means that $\gamma:V\longrightarrow T(V)/\mathcal{I}$ is a Clifford mapping. Therefore, $T(V)/\mathcal{I}$ is a Clifford algebra for (V,g). Right, let us prove the universality condition establish by the Definition 1.3 for $T(V)/\mathcal{I}$. Suppose that there is another Clifford algebra $(\mathcal{B},\rho)$ for the same quadratic space (V,g). For $(\mathcal{B},\rho)$ the function $\rho:V\longrightarrow\mathcal{B}$ such that $\rho(\mathbf{v})^2=Q(\mathbf{v})$ is considered. Since $\mathcal{L}(T_k(V),\mathcal{B})\simeq\mathcal{L}_{(k)}(V,\cdots,V;\mathcal{B})$ this mapping ρ can be extended to $T(V)$ as the linear map $\rho':T(V)\to\mathcal{B}$ given by $\rho=\rho'\circ\iota$

$$\begin{tikzcd} V \arrow[r, hook, "\iota"] \arrow[d, "\rho"'] & T(V) \arrow[dl, dashed, "\rho'"] \\ \mathcal{B} & \end{tikzcd}$$

such that

$$\rho'(\mathbf{v}_1\otimes\cdots\otimes\mathbf{v}_k)=(\rho'\circ\iota)(\mathbf{v}_1)\cdots(\rho'\circ\iota)(\mathbf{v}_k)=\rho(\mathbf{v}_1)\cdots\rho(\mathbf{v}_k). \tag{1.17}$$

Since $\rho(\mathbf{v})^2=Q(\mathbf{v})$, we have

$$\rho'(\mathbf{v}\otimes\mathbf{v}-Q(\mathbf{v})1_T)=0. \tag{1.18}$$

Moreover, we also have that since elements of that type $\mathbf{v}\otimes\mathbf{v}-Q(\mathbf{v})1_T$ are the generators of the two-sided ideal $\mathcal{I}$ it follows that $\mathcal{I}\subset\ker\rho'$. Let us consider then $T(V)/\ker\rho'$. In this case there exists a map $\phi:T(V)/\ker\rho'\to\mathcal{B}$ such that for all $x\in T(V)$

$$\phi([x]) = \rho'(x). \tag{1.19}$$

Which is an homomorphism

$$\phi([x][y]) = \phi([x \otimes y] = \rho'(x \otimes y) = \rho'(x)\rho'(y) = \phi([x])\phi([y]). \tag{1.20}$$

On the other hand, we have that if U_i is a subspace of a vector space $W_i\,(i = 1,2)$, and if $f : W_1 \to W_2$ is a linear map such that $f(U_1) \subset U_2$, then f induces a linear map $f' : W_1/U_1 \to W_2/U_2$. Moreover, when f is surjective, then f' is surjective too. Therefore, the surjective homomorphism $T(V)/\mathcal{I} \to T(V)/\mathrm{ker}\rho'$ shows that dim $T(V)/\mathcal{I} \geq$ dim $T(V)/\mathrm{ker}\rho'$. The homomorphism $T(V)/\mathcal{I} \to \mathcal{B}$ follows from $T(V)/\mathcal{I} \to T(V)/\mathrm{ker}\rho' \to \mathcal{B}$, say ϕ', then

$$\begin{tikzcd}
V \arrow[r, hook, "\iota"] \arrow[rr, bend left, "\gamma"] \arrow[d, "\rho"'] & T(V) \arrow[r, "\pi"] \arrow[dl, dashed, "\rho'"'] & T(V)/\mathcal{I} \arrow[dll, dashed, "\phi'"] \\
\mathcal{B} & &
\end{tikzcd}$$

Hence,

$$\begin{aligned}
\phi' \circ \pi &= \rho' \\
\phi' \circ (\pi \circ \iota) &= \rho' \circ \iota \\
\phi' \circ \gamma &= \rho
\end{aligned} \tag{1.21}$$

as desired. It is straightforward to show the uniqueness of ϕ' since if there were $\phi'_1 \circ \gamma = \rho$ and $\phi'_2 \circ \gamma = \rho$ they would coincide on the vector space V the generator of $T(V)$ and thereby on $T(V)/\mathcal{I}$ then ϕ_1, ϕ_2 would be the same. Finally, we conclude that $T(V)/\mathcal{I}$ is the universal Clifford algebra for the quadratic space (V, g), i.e., $T(V)/\mathcal{I} = \mathcal{C}\ell(V, g)$. $\square$

The homomorphism between the Clifford algebras $\phi' : \mathcal{C}\ell(V,g) = T(V)/\mathcal{I} \to \mathcal{B}$ defined at the end of the proof of Theorem 1.5 is injective and even bijective if $\mathcal{B}$ is generated by $\rho(V)$, then in this case they are isomorphic. For simplicity the word *universal* will be suppressed henceforward.

Right, since for any quadratic space it is possible to define a Clifford algebra, let g be a symmetric bilinear form in $\mathbb{R}^n$ of signature (p,q) where $p + q = n$ and let

$\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be an orthonormal basis of such vector space. The symmetric bilinear form can be evaluated on a vector $\mathbf{v} = v^i \mathbf{e}_i \in \mathbb{R}^n$ and yields that

$$g(\mathbf{v}, \mathbf{v}) = (v^1)^2 + \cdots + (v^p)^2 - (v^{p+1})^2 - \cdots - (v^{p+q})^2. \tag{1.22}$$

We denote this quadratic space by $\mathbb{R}^{p,q}$. It follows from Theorem 1.5 that the corresponding Clifford algebra of $\mathbb{R}^{p,q}$ exists and it is denoted by $\mathcal{C}\ell_{p,q}$:

$$\mathcal{C}\ell_{p,q} = \mathcal{C}\ell(\mathbb{R}^{p,q}).$$

We will now present a result that follows from the previous Theorem 1.5 that gives us the expression for the Clifford product on the Clifford algebras [7].

Corollary 1.6. *For $\mathbf{v} \in V$, $A_{[p]} \in \bigwedge_p(V)$, if $[\mathbf{v}], [A_{[p]}] \in T(V)/\mathcal{I} = \mathcal{C}\ell(V, g)$ then the Clifford product, denoted by juxtaposition, is given by*

$$\mathbf{v}A_{[p]} = \mathbf{v} \wedge A_{[p]} + \mathbf{v}_\flat \lrcorner A_{[p]} \tag{1.23}$$

where $\wedge$ is the exterior product, $\lrcorner$ is the left contraction both presented in the Appendix C, and $\mathbf{v}_\flat$ is such that $\flat$ is the musical isomorphism presented in the Appendix A.

Proof. It yields from Eq. (1.15) that the product of vectors $\mathbf{v}, \mathbf{u}$ in the quotient algebra $T(V)/\mathcal{I} = \mathcal{C}\ell(V, g)$, without the bracket notation, is given by

$$\mathbf{v}\mathbf{u} = \mathbf{v} \wedge \mathbf{u} + g(\mathbf{v}, \mathbf{u}). \tag{1.24}$$

The next step is to generalise the above expression (1.24) considering $\mathbf{v}, \mathbf{u}, \mathbf{w} \in V$. The exterior product $\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w}$ can be written as

$$\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w} = \frac{1}{3}(\mathbf{v} \otimes (\mathbf{u} \wedge \mathbf{w}) - \mathbf{u} \otimes (\mathbf{v} \wedge \mathbf{w}) + \mathbf{w} \otimes (\mathbf{v} \wedge \mathbf{u})). \tag{1.25}$$

By Eq. (1.15) we have that

$$\begin{aligned} \mathbf{v} \otimes \mathbf{u} &\sim \mathbf{v} \wedge \mathbf{u} + g(\mathbf{v}, \mathbf{u}) \\ \mathbf{w} \otimes (\mathbf{v} \otimes \mathbf{u}) &\sim \mathbf{w} \otimes (\mathbf{v} \wedge \mathbf{u} + g(\mathbf{v}, \mathbf{u})) \\ \mathbf{w} \otimes \mathbf{v} \otimes \mathbf{u} &\sim \mathbf{w} \otimes (\mathbf{v} \wedge \mathbf{u}) + g(\mathbf{v}, \mathbf{u})\mathbf{w} \\ \implies \mathbf{w} \otimes (\mathbf{v} \wedge \mathbf{u}) &\sim \mathbf{w} \otimes \mathbf{v} \otimes \mathbf{u} - g(\mathbf{v}, \mathbf{u})\mathbf{w}. \end{aligned} \tag{1.26}$$

Hence, we have the following relations

$$\begin{aligned} \mathbf{u}\otimes(\mathbf{v}\wedge\mathbf{w}) &\sim \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w} - g(\mathbf{v},\mathbf{w})\mathbf{u}, \\ \mathbf{w}\otimes(\mathbf{v}\wedge\mathbf{u}) &\sim \mathbf{w}\otimes\mathbf{v}\otimes\mathbf{u} - g(\mathbf{v},\mathbf{u})\mathbf{w}. \end{aligned} \tag{1.27}$$

Since $\wedge$ is antisymmetric it follows from Eq. (1.15) that

$$\begin{aligned} \mathbf{u}\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u}\sim 2g(\mathbf{u},\mathbf{v}) &\implies (\mathbf{u}\otimes\mathbf{v})\otimes\mathbf{w}+(\mathbf{v}\otimes\mathbf{u})\otimes\mathbf{w}\sim 2g(\mathbf{u},\mathbf{v})\mathbf{w}, \\ \mathbf{w}\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{w}\sim 2g(\mathbf{w},\mathbf{v}) &\implies (\mathbf{w}\otimes\mathbf{v})\otimes\mathbf{u}+(\mathbf{v}\otimes\mathbf{w})\otimes\mathbf{u}\sim 2g(\mathbf{w},\mathbf{v})\mathbf{u}. \end{aligned} \tag{1.28}$$

By using the above relations (1.28) in Eq. (1.27) it yields

$$\begin{aligned} \mathbf{u}\otimes(\mathbf{v}\wedge\mathbf{w}) &\sim -\mathbf{v}\otimes\mathbf{u}\otimes\mathbf{w} + 2g(\mathbf{u},\mathbf{v})\mathbf{w} - g(\mathbf{v},\mathbf{w})\mathbf{u}, \\ \mathbf{w}\otimes(\mathbf{v}\wedge\mathbf{u}) &\sim -\mathbf{v}\otimes\mathbf{w}\otimes\mathbf{u} + 2g(\mathbf{w},\mathbf{v})\mathbf{u} - g(\mathbf{v},\mathbf{u})\mathbf{w}. \end{aligned} \tag{1.29}$$

Using the relations written in Eq. (1.27) we get

$$\begin{aligned} \mathbf{v}\otimes\mathbf{u}\otimes\mathbf{w} &\sim \mathbf{v}\otimes(\mathbf{u}\wedge\mathbf{w}) + g(\mathbf{u},\mathbf{w})\mathbf{w}, \\ -\mathbf{v}\otimes\mathbf{w}\otimes\mathbf{u} &\sim -\mathbf{v}\otimes(\mathbf{w}\wedge\mathbf{u}) - g(\mathbf{w},\mathbf{u})\mathbf{v} = \mathbf{v}\otimes(\mathbf{u}\wedge\mathbf{w}) - g(\mathbf{w},\mathbf{u})\mathbf{v}. \end{aligned} \tag{1.30}$$

Substituting the Eqs. (1.29) and (1.30) in Eq. (1.25) we obtain

$$\begin{aligned} \mathbf{v}\wedge\mathbf{u}\wedge\mathbf{w} &= \frac{1}{3}(\mathbf{v}\otimes(\mathbf{u}\wedge\mathbf{w}) - \mathbf{u}\otimes(\mathbf{v}\wedge\mathbf{w}) + \mathbf{w}\otimes(\mathbf{v}\wedge\mathbf{u})) \\ &\sim \frac{1}{3}(\mathbf{v}\otimes(\mathbf{u}\wedge\mathbf{w}) + \mathbf{v}\otimes\mathbf{u}\otimes\mathbf{w} - 2g(\mathbf{u},\mathbf{v})\mathbf{w} + g(\mathbf{v},\mathbf{w})\mathbf{u} \\ &\quad - \mathbf{v}\otimes\mathbf{w}\otimes\mathbf{u} + 2g(\mathbf{w},\mathbf{v})\mathbf{u} - g(\mathbf{v},\mathbf{u})\mathbf{w} \\ &\sim \frac{1}{3}(\mathbf{v}\otimes(\mathbf{u}\wedge\mathbf{w}) + \mathbf{v}\otimes(\mathbf{u}\wedge\mathbf{w}) + g(\mathbf{u},\mathbf{w})\mathbf{w} + \mathbf{v}\otimes(\mathbf{u}\wedge\mathbf{w}) - g(\mathbf{w},\mathbf{u})\mathbf{v} \\ &\quad + 3g(\mathbf{v},\mathbf{w})\mathbf{u} - 3g(\mathbf{u},\mathbf{v})\mathbf{w} \\ &\sim \mathbf{v}\otimes(\mathbf{u}\wedge\mathbf{w}) + g(\mathbf{v},\mathbf{w})\mathbf{u} - g(\mathbf{v},\mathbf{u})\mathbf{w}. \end{aligned} \tag{1.31}$$

Recalling the musical isomorphism flat $\flat$ presented in the Eq. (A.11) and the contraction expression for bivectors stated in Eq. (C.46), we have that

$$g(\mathbf{v},\mathbf{u})\mathbf{w} - g(\mathbf{v},\mathbf{w})\mathbf{u} = -\mathbf{v}_\flat(\mathbf{u})\mathbf{w} - \mathbf{v}_\flat(\mathbf{w})\mathbf{u} = \mathbf{v}_\flat\rfloor(\mathbf{u}\wedge\mathbf{w}), \tag{1.32}$$

which implies that the product of a vector and a bivector on $\mathcal{C}\ell(V,g)$ can be written as

$$\mathbf{v}(\mathbf{u}\wedge\mathbf{w}) = \mathbf{v}\wedge\mathbf{u}\wedge\mathbf{w} + \mathbf{v}_\flat\rfloor(\mathbf{u}\wedge\mathbf{w}). \tag{1.33}$$

By induction, one may generalise the Clifford product of a vector $\mathbf{v}$ and a p-vector $A_{[p]}$ as [7]

$$\mathbf{v}A_{[p]} = \mathbf{v} \wedge A_{[p]} + \mathbf{v}_\flat \rfloor A_{[p]}, \tag{1.34}$$

which proves the corollary. □

Similarly, the Clifford product of a p-vector $A_{[p]}$ and a vector $\mathbf{v}$ is given by

$$A_{[p]}\mathbf{v} = A_{[p]} \wedge \mathbf{v} + A_{[p]} \lfloor \mathbf{v}_\flat. \tag{1.35}$$

Such that [7]

$$\begin{aligned} A_{[p]} \wedge \mathbf{v} &= (-1)^p \mathbf{v} \wedge A_{[p]}, \\ A_{[p]} \lfloor \mathbf{v}_\flat &= -(-1)^p \mathbf{v}_\flat \rfloor A_{[p]}. \end{aligned} \tag{1.36}$$

Therefore, we have

$$A_{[p]}\mathbf{v} = (-1)^p \mathbf{v} \wedge A_{[p]} - (-1)^p \mathbf{v}_\flat \rfloor A_{[p]}. \tag{1.37}$$

Now, comparing the equations (1.34) and (1.35) it follows that

$$\begin{aligned} \mathbf{v} \wedge A_{[p]} &= \frac{1}{2}(\mathbf{v}A_{[p]} + (-1)^p A_{[p]}\mathbf{v}), \\ \mathbf{v}_\flat \lfloor A_{[p]} &= \frac{1}{2}(\mathbf{v}A_{[p]} - (-1)^p A_{[p]}\mathbf{v}). \end{aligned} \tag{1.38}$$

1.3. Examples

Example 1.7 ▹ $\mathbb{C}$

Let V be a 1-dimensional vector space with basis $\{\mathbf{e}\}$, i.e., for all $\mathbf{v} \in V, \mathbf{v} = y\mathbf{e}$. Consider g the symmetric bilinear functional such that $g(\mathbf{e},\mathbf{e}) = -1$, then, for all $\mathbf{v} \in V, g(\mathbf{v},\mathbf{v}) = -y^2$. Consider now the subalgebra $\mathcal{A}$ of the matrix algebra $\mathcal{M}(2,\mathbb{R})$ defined by

$$\mathcal{A} = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix}; x, y \in \mathbb{R} \right\}. \tag{1.39}$$

We have that $\mathcal{A}$ is generated by

$$\left\{ y\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; y \in \mathbb{R} \right\} \quad \text{and} \quad \left\{ x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = x1_{\mathcal{A}}; x \in \mathbb{R} \right\}. \tag{1.40}$$

Continuing in this way, define $\gamma : V \to \mathcal{A}$ as

$$\gamma(\mathbf{e}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \implies \gamma(\mathbf{v}) = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}, \tag{1.41}$$

thus

$$\gamma(\mathbf{v})^2 = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}\begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix} = -y^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g(\mathbf{v},\mathbf{v})1_{\mathcal{A}}, \tag{1.42}$$

which means that γ is a Clifford mapping. Considering the following isomorphism

$$\mathbb{C} \leftrightarrow \mathcal{M}(2,\mathbb{R}), \quad x + iy \leftrightarrow \begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \tag{1.43}$$

We conclude that $\mathcal{A} \simeq \mathbb{C}$, therefore $\mathbb{C}$ is an example of a Clifford algebra. ◃

Example 1.8 ▹ $\mathcal{M}(2,\mathbb{C})$

Let us consider $V = \mathbb{R}^3$, the canonical basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $g(\mathbf{v},\mathbf{u}) = \langle \mathbf{v},\mathbf{u}\rangle$ the usual scalar product in $\mathbb{R}^3$. Consider $\mathcal{A} = \mathcal{M}(2,\mathbb{C})$ the algebra of complex matrix 2×2

$$\mathcal{A} = \left\{ \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}; z_i \in \mathbb{C} \right\}. \tag{1.44}$$

Let us consider now the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.45}$$

We have that $\mathcal{A}$ is generated by $\{x_1\sigma_1, x_2\sigma_2, x_3\sigma_3 \mid x_i \in \mathbb{C}\}$ and $\{x_0 I \mid x_0 \in \mathbb{C}\}$. Define the mapping $\gamma : V \to \mathcal{A}$ as

$$\gamma(\mathbf{e}_1) = \sigma_1, \gamma(\mathbf{e}_2) = \sigma_2, \gamma(\mathbf{e}_3) = \sigma_3. \tag{1.46}$$

It follows that $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3, \mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3, \in \mathbb{R}^3$

$$\begin{aligned} \gamma(\mathbf{v}) &= \gamma(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 \\ &= \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} \\ &\implies \gamma(\mathbf{v})\gamma(\mathbf{u}) + \gamma(\mathbf{u})\gamma(\mathbf{v}) = 2(v_1u_1 + v_2u_2 + v_3u_3)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2g(\mathbf{v},\mathbf{u})1_{\mathcal{A}}. \end{aligned} \tag{1.47}$$

Therefore, γ is a Clifford mapping and $\mathcal{A} = \mathcal{M}(2,\mathbb{C})$ is a Clifford algebra. ◃

Example 1.9 ▹ $\mathcal{C}\ell_3$

Let us consider $V = \mathbb{R}^3$ endowed with the usual scalar product. The universal Clifford algebra of such quadratic space is denoted by $\mathcal{C}\ell_3$ and is generated by $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, with dimension $2^3 = 8$ such that

$$\begin{aligned} \mathcal{C}\ell_3 &= \operatorname{span}\{\gamma(\mathbf{e}_1)^{\mu_1}\gamma(\mathbf{e}_2)^{\mu_2}\gamma(\mathbf{e}_3)^{\mu_3} \mid \mu_i = 0, 1\} \\ &= \operatorname{span}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}. \end{aligned} \tag{1.48}$$

and satisfies

$$\mathbf{e_i}\mathbf{e_j} + \mathbf{e_j}\mathbf{e_i} = 2\delta_{ij}1. \tag{1.49}$$

Such that δ_{ij} is the Kronecker delta. The Clifford product between two vectors $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ and $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ is given by

$$\begin{aligned} \mathbf{uv} &= u_1v_1 + u_2v_2 + u_3v_3 + (u_2v_3 - u_3v_2)\mathbf{e}_{23} \\ &\quad + (u_3v_1 - u_1v_3)\mathbf{e}_{31} + (u_1v_2 - u_2v_1)\mathbf{e}_{12} \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}. \end{aligned} \tag{1.50}$$

One may also denote a multivector $\mathbf{e}_i\mathbf{e}_j$ as $\mathbf{e}_{ij}$ where convenient for simplicity.

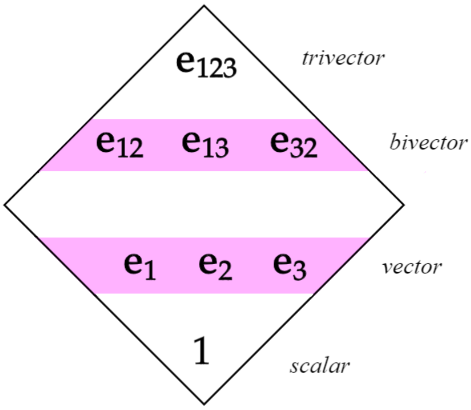

Figure 1.1.: The multivector structure of $\mathcal{C}\ell_3$

As we expect from the previous results, both Clifford algebras $\mathcal{M}(2, \mathbb{C})$ and the universal Clifford algebra $\mathcal{C}\ell_3$ are isomorphic and an element of $\mathcal{C}\ell_3$ can be represented as a matrix on $\mathcal{M}(2, \mathbb{C})$ through the identification $\mathbf{e}_1 \leftrightarrow \sigma_1$, $\mathbf{e}_2 \leftrightarrow \sigma_2$, $\mathbf{e}_3 \leftrightarrow \sigma_3$. A detailed discussion about Clifford algebra and its representations will be presented later on. ◃

Example 1.10 ▹ $\mathcal{C}\ell_{1,3}$

Let $V = \mathbb{R}^{1,3}$ be the Minkowski space-time. Considering an orthonormal basis of such space as $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathbf{v} = (v_0, v_1, v_2, v_3)$ an arbitrary vector, on this 4-dimensional real space endowed with the metric g, it follows that

$$\begin{aligned} g(\mathbf{v},\mathbf{v}) &= (v_0\mathbf{e}_0 + v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3)(v_0\mathbf{e}_0 + v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\ &= v_o^2(\mathbf{e}_0)^2 + v_1^2(\mathbf{e}_1)^2 + v_2^2(\mathbf{e}_2)^2 + v_3^2(\mathbf{e}_3)^2 \\ &\quad + v_0v_1(\mathbf{e}_0\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_0) + v_0v_2(\mathbf{e}_0\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_0) + v_0v_3(\mathbf{e}_0\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_0) \\ &\quad + v_1v_2(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) + v_1v_3(\mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1) + v_2v_3(\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2) \\ &= v_0^2 - v_1^2 - v_2^2 - v_3^2. \end{aligned}$$

The Clifford algebra of such space is denoted by $\mathcal{C}\ell_{1,3}$ and is generated by

$$\begin{aligned} \mathcal{C}\ell_{1,3} &= \text{span}\{\gamma(\mathbf{e}_0)^{\mu_0}\gamma(\mathbf{e}_1)^{\mu_1}\gamma(\mathbf{e}_2)^{\mu_2}\gamma(\mathbf{e}_3)^{\mu_3} \mid \mu_i = 0,1\} \\ &= \text{span}\{1, \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_0\mathbf{e}_1, \mathbf{e}_0\mathbf{e}_2, \mathbf{e}_0\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2, \\ &\quad \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_0\mathbf{e}_1\mathbf{e}_3, \mathbf{e}_0\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}. \end{aligned} \tag{1.51}$$

The basis elements satisfy the following relations

$$\begin{aligned} &(\mathbf{e}_0)^2 = 1, \\ &(\mathbf{e}_i)^2 = -1, \qquad\qquad i = 1,2,3. \\ &(\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu) = 0, \quad \mu \neq \nu \quad \mu,\nu = 0,1,2,3. \end{aligned} \tag{1.52}$$

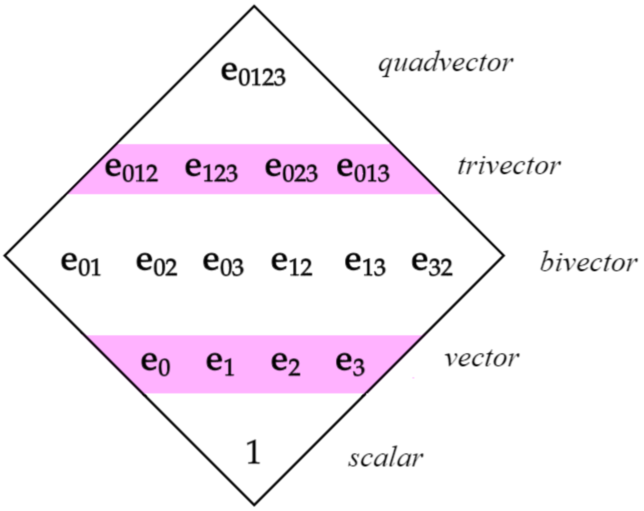

Figure 1.2.: The multivector structure of $\mathcal{C}\ell_{1,3}$

An arbitrary element A of $\mathcal{C}\ell_{1,3}$ is written as

$$\begin{aligned} A = {} & a + a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{01}\mathbf{e}_0\mathbf{e}_1 + a_{02}\mathbf{e}_0\mathbf{e}_2 \\ & + a_{03}\mathbf{e}_0\mathbf{e}_3 + a_{12}\mathbf{e}_1\mathbf{e}_2 + a_{13}\mathbf{e}_1\mathbf{e}_3 + a_{23}\mathbf{e}_2\mathbf{e}_3 + a_{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 \\ & + a_{013}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3 + a_{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + a_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + a_{0123}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3. \quad \triangleleft \end{aligned} \tag{1.53}$$

Let us consider the exterior algebra defined in the Appendix C.

Proposition 1.11. *The exterior algebra is a Clifford algebra.*

Proof. Let V be a vector space and $(\bigwedge(V), \wedge)$ be the exterior algebra. Let us consider the mapping $\gamma : V \to \bigwedge^1(V)$. For arbitrary elements $\mathbf{u}, \mathbf{v} \in V$ it reads that $\gamma(\mathbf{v})\gamma(\mathbf{u}) = \mathbf{v} \wedge \mathbf{u}$. We also have that each element of $\bigwedge^k(V)$ is given by $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k = \gamma(\mathbf{v}_1)\ldots\gamma(\mathbf{v}_k)$. Therefore, it follows that $\bigwedge(V)$ is generated by $\{\gamma(\mathbf{v}) \mid \mathbf{v} \in V\}$ e $\{x1_{\mathcal{A}} \mid x \in \mathbb{R}\}$. For the symmetric bilinear form $g = 0$, we have:

$$\gamma(\mathbf{v})\gamma(\mathbf{u}) + \gamma(\mathbf{u})\gamma(\mathbf{v}) = \mathbf{v} \wedge \mathbf{u} + \mathbf{u} \wedge \mathbf{v} = 0 = 2g(\mathbf{u}, \mathbf{v})1_{\mathcal{A}}. \tag{1.54}$$

Therefore the exterior algebra is a Clifford algebra with respect to the null bilinear form $g = 0$. □

2 STRUCTURE OF CLIFFORD ALGEBRAS

This Chapter provides a detailed exposition of the Clifford algebra structure. We will present some operations that can be defined on those algebras, some important subspaces and as the main goal of this Chapter: the main theorems concerning the Clifford algebras structure that unravel some important behaviour of these algebras like its periodicity, that allow us in the next Chapter, present the objective of this work: the Clifford algebras classification.

2.1. General Properties

Let V be a vector space and $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be an orthogonal basis for V. It yields from Eq. (1.34) that

$$\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j + (\mathbf{e}_i)_\flat \lrcorner \mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j + (\mathbf{e}_i)_\flat(\mathbf{e}_j) = \mathbf{e}_i \wedge \mathbf{e}_j + g(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{e}_i \wedge \mathbf{e}_j \tag{2.1}$$

for every $i \neq j$. One may generalise the above result and obtain the following relation

$$\mathbf{e}_{\mu_1} \cdots \mathbf{e}_{\mu_p} = \mathbf{e}_{\mu_1} \wedge \cdots \wedge \mathbf{e}_{\mu_p} \qquad (\mu_1 \neq \cdots \neq \mu_p). \tag{2.2}$$

Hence, as it is derived in the Eq. (C.24), the dimension of $\mathcal{C}\ell(V, g)$ is given by

$$\dim \mathcal{C}\ell(V, g) = \sum_{p=0}^{n} \binom{n}{p} = 2^n = 2^{\dim V}. \tag{2.3}$$

We have therefore the converse of Theorem 1.4,

> every universal Clifford algebra has dimension 2^n ($n = \dim V$).

There exists a *vector space isomorphism* between the exterior algebra $\bigwedge(V)$ and the Clifford algebra $\mathcal{C}\ell(V, g)$ since they have the same dimension. One important point to note here is the fact that Clifford algebra inherits the multivector structure from exterior algebra. One may write

$$\mathcal{C}\ell(V,g) = \bigoplus_{p=0}^{n} \bigwedge_{p}(V). \tag{2.4}$$

This means that the study of the exterior algebra is fundamental to understand the multivector structure of the Clifford algebras. A discussion of the exterior algebra can be found in the Appendix C.

2.2. Operations on Clifford Algebras

Some operations can be defined in a multivector algebra based on its own multivector structure. They can be used to unravel some properties in the algebraic structure, to define some subspaces of the algebra and its applications. Those operations can be defined equivalently with respect to the exterior algebra $\bigwedge(V)$ or in terms of Clifford algebras $\mathcal{C}\ell(V,g)$.

Definition 2.1. *Let* $A \in \mathcal{C}\ell(V,g)$ *be a multivector, the* ***projector operator*** *is defined as*

$$\begin{aligned} \langle \cdot \rangle_p : \mathcal{C}\ell(V,g) &\longrightarrow \bigwedge_{p}(V) \\ \langle A \rangle_p &\longmapsto A_{[p]}, \end{aligned} \tag{2.5}$$

such that $A_{[p]} \in \bigwedge_p(V)$ *is the p-vector component of A.*

Definition 2.2. *Let* $A \in \mathcal{C}\ell(V,g)$ *be a multivector, the* ***grade involution*** *is defined as*

$$\begin{aligned} \widehat{\cdot} : \mathcal{C}\ell(V,g) &\longrightarrow \mathcal{C}\ell(V,g) \\ \widehat{A_{[p]}} &\longmapsto (-1)^p A_{[p]}. \end{aligned} \tag{2.6}$$

Definition 2.3. *Let* $A \in \mathcal{C}\ell(V,g)$ *be a multivector, the* ***reversion*** *is defined as follows*

$$\begin{aligned} \widetilde{\cdot} : \mathcal{C}\ell(V,g) &\longrightarrow \mathcal{C}\ell(V,g) \\ \widetilde{A_{[p]}} &\longmapsto (-1)^{\frac{p(p-1)}{2}} A_{[p]}. \end{aligned} \tag{2.7}$$

Definition 2.4. *Let* $A \in \mathcal{C}\ell(V,g)$ *be a multivector, the* ***conjugation****, denoted by* $\overline{A}$*, is defined as the composition of the grade involution and the grade involution:*

$$\widehat{\widetilde{A_{[p]}}} = \widetilde{\widehat{A_{[p]}}} = \overline{A_{[p]}}. \tag{2.8}$$

In summary, for $A \in \mathcal{C}\ell(V,g)$:

(i) $\langle A\rangle_p = A_{[p]}$, (projection);

(ii) $\widehat{A_{[p]}} = (-1)^p A_{[p]}$, (grade involution);

(iii) $\widetilde{A_{[p]}} = (-1)^{\frac{p(p-1)}{2}} A_{[p]}$, (reversion);

(iv) $\overline{A_{[p]}} = \widetilde{\widehat{A_{[p]}}} = \widehat{\widetilde{A_{[p]}}}$, (conjugation).

Example 2.5 ▹ Operations in $\mathcal{C}\ell_{1,3}$

Let us consider the Clifford algebra $\mathcal{C}\ell_{1,3}$ and an arbitrary $A \in \mathcal{C}\ell_{1,3}$ given by

$$\begin{aligned} A = {} & a + a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{01}\mathbf{e}_0\mathbf{e}_1 + a_{02}\mathbf{e}_0\mathbf{e}_2 \\ & + a_{03}\mathbf{e}_0\mathbf{e}_3 + a_{12}\mathbf{e}_1\mathbf{e}_2 + a_{13}\mathbf{e}_1\mathbf{e}_3 + a_{23}\mathbf{e}_2\mathbf{e}_3 + a_{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 \\ & + a_{013}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3 + a_{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + a_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + a_{0123}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3. \end{aligned} \tag{2.9}$$

One may also write A in a compact form

$$A = A_{[0]} + A_{[1]} + A_{[2]} + A_{[3]} + A_{[4]}. \tag{2.10}$$

The projector operator yields:

$$\begin{aligned} &\langle A\rangle_0 = a = A_{[0]} \in \bigwedge^0(\mathbb{R}^{1,3}), \\ &\langle A\rangle_1 = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = A_{[1]} \in \bigwedge^1(\mathbb{R}^{1,3}), \\ &\langle A\rangle_2 = a_{01}\mathbf{e}_0\mathbf{e}_1 + a_{02}\mathbf{e}_0\mathbf{e}_2 + a_{03}\mathbf{e}_0\mathbf{e}_3 + a_{12}\mathbf{e}_1\mathbf{e}_2 + a_{13}\mathbf{e}_1\mathbf{e}_3 + a_{23}\mathbf{e}_2\mathbf{e}_3 = A_{[2]} \in \bigwedge^2(\mathbb{R}^{1,3}), \\ &\langle A\rangle_3 = a_{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + a_{013}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3 + a_{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + a_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = A_{[3]} \in \bigwedge^3(\mathbb{R}^{1,3}). \\ &\langle A\rangle_4 = a_{0123}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = A_{[4]} \in \bigwedge^4(\mathbb{R}^{1,3}). \end{aligned} \tag{2.11}$$

For the other operations we have that

$$\begin{aligned} \widehat{A} &= A_{[0]} - A_{[1]} + A_{[2]} - A_{[3]} + A_{[4]}, && \text{grade involution,} \\ \widetilde{A} &= A_{[0]} + A_{[1]} - A_{[2]} - A_{[3]} + A_{[4]}, && \text{reversion,} \\ \overline{A} &= A_{[0]} - A_{[1]} - A_{[2]} + A_{[3]} + A_{[4]}, && \text{conjugation.} \end{aligned} \tag{2.12}$$ ◃

Proposition 2.6. *Let* $A_{[p]} \in \bigwedge_p(V)$*, then* $\widehat{A_{[p]}} = \pm A_{[p]}$*,* $\widetilde{A_{[p]}} = \pm A_{[p]}$ *and* $\overline{A_{[p]}} = \pm A_{[p]}$ *with the plus or minus sign depending only on* $p \mod 4$ *as indicated in the following table:*

$p \mod 4$	0	1	2	3
grade involution $(\widehat{\cdot})$	+	−	+	−
reversion $(\widetilde{\cdot})$	+	+	−	−
conjugation $(\overline{\cdot})$	+	−	−	+

Proof. For the grade involution it is straightforward since $\widehat{A_{[p]}} = (-1)^p A_{[p]}$. For the reversion, we have that $\widetilde{A_{[p]}} = (-1)^{\frac{p(p-1)}{2}} A_{[p]}$. Therefore, let us consider the four cases for $p \in \mathbb{N}$. Suppose that $p = 0 \mod 4$. In this case, $p = 4x$ for some $x \in \mathbb{N}$, therefore $\frac{1}{2}p(p-1) = \frac{1}{2}4x(4x-1)$ which is an even number. Suppose that $p = 1 \mod 4$. In this case, $p = 4x+1$, hence $\frac{1}{2}p(p-1) = \frac{1}{2}(4x+1)(4x)$ which is an even number. Now let $p = 2 \mod 4$, therefore $\frac{1}{2}p(p-1) = \frac{1}{2}(4x+2)(4x-1)$ which is an odd number. Finally, for $p = 3 \mod 4$ it follows that $\frac{1}{2}p(p-1) = \frac{1}{2}(4x+3)(4x-2)$ which is an odd number. Right, once the conjugation is the composition of the grade involution and the reversion, so is its signal. □

2.3. Subspaces of Clifford Algebras

Let us now introduce some important subspaces of $\mathcal{C}\ell(V,g)$.

Definition 2.7. *The* ***centre*** *of a Clifford algebra* $\mathcal{C}\ell_{p,q}$ *is defined as being the set of elements on* $\mathcal{C}\ell_{p,q}$ *that commutes with all elements of* $\mathcal{C}\ell_{p,q}$

$$\mathrm{Cen}(\mathcal{C}\ell_{p,q}) = \{a \in \mathcal{C}\ell_{p,q} \mid ax = xa, \forall\, x \in \mathcal{C}\ell_{p,q}\}. \tag{2.13}$$

One may characterise the $\mathrm{Cen}(\mathcal{C}\ell_{p,q})$, $n = p+q$, as [7]

$$\mathrm{Cen}(\mathcal{C}\ell_{p,q}) = \begin{cases} \bigwedge_0(\mathbb{R}^{p,q}), \text{ if } n \text{ is even,} \\ \bigwedge_0(\mathbb{R}^{p,q}) \oplus \bigwedge_n(\mathbb{R}^{p,q}), \text{ if } n \text{ is odd.} \end{cases} \tag{2.14}$$

The next definition allows us to write the Clifford algebra $\mathcal{C}\ell(V,g)$ as a direct sum of two subspaces.

Definition 2.8. *The following subspaces of* $\mathcal{C}\ell(V,g)$

$$\begin{aligned} \mathcal{C}\ell^+(V,g) &= \{A \in \mathcal{C}\ell(V,g) \mid \widehat{A} = A\}, \\ \mathcal{C}\ell^-(V,g) &= \{A \in \mathcal{C}\ell(V,g) \mid \widehat{A} = -A\}, \end{aligned} \tag{2.15}$$

are said to be respectively the ***even part*** *and the* ***odd part*** *of* $\mathcal{C}\ell(V,g)$.

Therefore, one can write

$$\mathcal{C}\ell(V,g) = \mathcal{C}\ell^{+}(V,g) \oplus \mathcal{C}\ell^{-}(V,g). \tag{2.16}$$

Those subspaces unravel one important property of the Clifford algebra structure concerning its grading.

Definition 2.9. *Let* G *be an abelian group. An algebra* $\mathcal{A}$ *is said to be* ***G-graded*** *if there exists subspaces* $\mathcal{A}_k$ $(k \in G)$ *such that* $\mathcal{A} = \bigoplus \mathcal{A}_k$ *and, if given* $x_k \in \mathcal{A}_k$, $y_l \in \mathcal{A}_l$ *it follows that* $x_k y_l \in \mathcal{A}_{k+l}$.

The elements of $\mathcal{A}_k$ are said to be *homogeneous of degree* k. In general, the following notation is used

$$k = \deg(x_k), \quad x_k \in \mathcal{A}_k. \tag{2.17}$$

Since G is abelian it implies that

$$\deg(x_k y_l) = \deg(x_k) + \deg(y_l). \tag{2.18}$$

It is assumed that for the scalars $a \in \mathbb{K} = \mathbb{C}, \mathbb{R}$ that $\deg(a) = 0$ and that the null vector must be considered homogeneous for all degrees since every subspace $\mathcal{A}_k$ contains it. If the unique element which is negative-graded is the null vector, the algebra is said to be positive-graded.

Right, since the grade involution is an automorphism [7]

$$\widehat{A_{[p]}B_{[q]}} = \widehat{A_{[p]}}\,\widehat{B_{[q]}} = (-1)^{p+q} A_{[p]}B_{[q]} \in \bigwedge\nolimits_{p+q}(V) \tag{2.19}$$

the following relations for the Clifford algebra $\mathcal{C}\ell(V,g)$ can be derived

$$\begin{aligned}
\mathcal{C}\ell^{+}(V,g)\mathcal{C}\ell^{+}(V,g) &\subset \mathcal{C}\ell^{+}(V,g),\\
\mathcal{C}\ell^{+}(V,g)\mathcal{C}\ell^{-}(V,g) &\subset \mathcal{C}\ell^{-}(V,g),\\
\mathcal{C}\ell^{-}(V,g)\mathcal{C}\ell^{+}(V,g) &\subset \mathcal{C}\ell^{-}(V,g),\\
\mathcal{C}\ell^{-}(V,g)\mathcal{C}\ell^{-}(V,g) &\subset \mathcal{C}\ell^{+}(V,g).
\end{aligned} \tag{2.20}$$

Thus, it follows that $\mathcal{C}\ell(V,g)$ is a $\mathbb{Z}_2$-graded algebra and $\mathcal{C}\ell^{+}(V,g)$ is a subalgebra of $\mathcal{C}\ell(V,g)$ called *even subalgebra*.

It is worth to emphasise that in contrast to the grade involution, we have that the reversion is an anti-automorphism, that is [8]

$$\widetilde{A_{[p]}B_{[q]}} = \widetilde{B_{[q]}}\,\widetilde{A_{[p]}}. \tag{2.21}$$

2.4. Theorems on Clifford Algebras Structure

In this section, some important theorems concerning the structure of Clifford algebras will be presented. Aside from the inherent importance of such theorems, they will be used to construct the classification of the Clifford algebras. Although, before presenting those theorems, let us start by defining an important concept regarding graded algebras.

Definition 2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be two graded algebras. The* ***alternating tensor product*** *$\mathcal{A}\hat{\otimes}\mathcal{B}$ between those algebras is defined as the algebra generated by the product $a\hat{\otimes}b$, $a\in\mathcal{A}$, $b\in\mathcal{B}$, with product defined by*

$$(a_1\hat{\otimes}b_1)(a_2\hat{\otimes}b_2) = (-1)^{\deg(b_1)\deg(a_2)}a_1a_2\hat{\otimes}b_1b_2. \tag{2.22}$$

The Definition 2.10 applies to homogeneous elements and can be extended by linearity. The tensor product $\hat{\otimes}$ is the usual tensor product and the hat notation reminds us that the product must take into account the grading of the elements involved. This leads to the following theorem [6].

Theorem 2.11. *Let (V,g) and (V',g') be two quadratic spaces and $\mathcal{C}\ell(V,g)$ and $\mathcal{C}\ell(V',g')$ their respective Clifford algebras. Thus*

$$\mathcal{C}\ell(V\oplus V', g\oplus g') \simeq \mathcal{C}\ell(V,g)\hat{\otimes}\mathcal{C}\ell(V',g') \tag{2.23}$$

where $\simeq$ denotes an isomorphism between Clifford algebras and $V\oplus V'$ stands for the orthogonal direct sum of V and V'.

Proof. Let $g\oplus g'$ be the symmetric bilinear form defined in $V\oplus V'$ such that for all $\mathbf{v},\mathbf{u}\in V$ and for all $\mathbf{v}',\mathbf{u}'\in V'$:

$$(g\oplus g')(\mathbf{v}+\mathbf{v}',\mathbf{u}+\mathbf{u}') = g(\mathbf{v},\mathbf{u}) + g'(\mathbf{v}',\mathbf{u}'). \tag{2.24}$$

Define a map $\Gamma : V\oplus V' \to \mathcal{C}\ell(V,g)\hat{\otimes}\mathcal{C}\ell(V',g')$ by setting

$$\Gamma(\mathbf{v}+\mathbf{v}') = \underbrace{\quad\mathbf{v}\quad}_{\gamma(\mathbf{v})}\hat{\otimes}\,1_{V'} + 1_V\,\hat{\otimes}\underbrace{\quad\mathbf{v}'\quad}_{\gamma'(\mathbf{v}')}. \tag{2.25}$$

Since $\gamma : V \to \mathcal{C}\ell(V,g)$ and $\gamma' : V' \to \mathcal{C}\ell(V',g')$ are Clifford mappings, then Γ is also a Clifford map. Indeed,

$$\begin{aligned}
(\Gamma(\mathbf{v}+\mathbf{v}'))^2 &= (\mathbf{v}\hat{\otimes} 1_{V'} + 1_V \hat{\otimes} \mathbf{v}')(\mathbf{v}\hat{\otimes} 1_{V'} + 1_V \hat{\otimes} \mathbf{v}') \\
&= (-1)^{0\cdot 1}\mathbf{v}^2 \hat{\otimes} 1_{V'} + (-1)^{0\cdot 0}\mathbf{v}\hat{\otimes}\mathbf{v}' + (-1)^{1\cdot 1}\mathbf{v}\hat{\otimes}\mathbf{v}' + (-1)^{1\cdot 0} 1_V \hat{\otimes} \mathbf{v}'^2 \\
&= \mathbf{v}^2 \hat{\otimes} 1_{V'} + \mathbf{v}\hat{\otimes}\mathbf{v}' - \mathbf{v}\hat{\otimes}\mathbf{v}' + 1_V \hat{\otimes} \mathbf{v}'^2 \\
&= g(\mathbf{v},\mathbf{v})\hat{\otimes} 1_{V'} + 1_V \hat{\otimes} g'(\mathbf{v}',\mathbf{v}') \qquad (2.26) \\
&= (g(\mathbf{v},\mathbf{v}))\, 1_V \hat{\otimes} 1_{V'} + (g'(\mathbf{v}',\mathbf{v}'))\, 1_V \hat{\otimes} 1_{V'} \\
&= (g(\mathbf{v},\mathbf{v}) + g'(\mathbf{v}',\mathbf{v}'))\, 1_V \hat{\otimes} 1_{V'} \\
&= (g \oplus g')(\mathbf{v}+\mathbf{v}', \mathbf{v}+\mathbf{v}')\, 1_V \hat{\otimes} 1_{V'}.
\end{aligned}$$

In addition,

$$\dim \mathcal{C}\ell(V,g)\hat{\otimes}\mathcal{C}\ell(V',g') = 2^{\dim V}\, 2^{\dim V'} = 2^{\dim V\oplus V'}. \qquad (2.27)$$

On the other hand, let $\mathcal{C}\ell(V\oplus V', g\oplus g')$ be the universal Clifford algebra associated to the quadratic space $(V\oplus V', g\oplus g')$, from the universality theorem, there exists a homomorphism

$$\phi : \mathcal{C}\ell(V\oplus V', g\oplus g') \to \mathcal{C}\ell(V,g)\hat{\otimes}\mathcal{C}\ell(V',g') \qquad (2.28)$$

such that the following diagram commutes

$$\begin{tikzcd}
V\oplus V' \arrow[r, "\gamma_{(V\oplus V')}"] \arrow[d, "\Gamma"'] & \mathcal{C}\ell(V\oplus V', g\oplus g') \arrow[dl, dashed, "\phi"] \\
\mathcal{C}\ell(V,g)\hat{\otimes}\mathcal{C}\ell(V',g') &
\end{tikzcd}$$

Where $\gamma_{(V\oplus V')}$ denotes the Clifford map. Therefore, since

$$\dim \mathcal{C}\ell(V\oplus V', g\oplus g') = 2^{\dim V\oplus V'} = \dim \mathcal{C}\ell(V,g)\hat{\otimes}\mathcal{C}\ell(V',g') \qquad (2.29)$$

we conclude that ϕ is an algebra isomorphism. □

Now we will introduce the concept regarding the *complexification* of a Clifford algebra, before, since an algebra is a vector space, let us state the complexification of a vector space first.

Definition 2.12. *Let V a vector space over $\mathbb{R}$ with dimension dim $V = n$. The* ***complexification*** *$V_{\mathbb{C}}$ is the space of elements in the form $\mathbf{v} + i\mathbf{u}$, such that $\mathbf{v}, \mathbf{u} \in V$ and i is the imaginary unit.*

Hence, the space $V_{\mathbb{C}}$ is a vector space with sum and multiplication by a complex scalar $(a + ib)$ defined by

$$\begin{aligned} (\mathbf{v}_1 + i\mathbf{u}_1) + (\mathbf{v}_2 + i\mathbf{u}_2) &= (\mathbf{v}_1 + \mathbf{v}_2) + i(\mathbf{u}_1 + \mathbf{u}_2), \\ (a + ib)(\mathbf{v} + i\mathbf{u}) &= (a\mathbf{v} - b\mathbf{u}) + i(b\mathbf{v} + a\mathbf{u}). \end{aligned} \tag{2.30}$$

The dimension of $V_{\mathbb{C}}$ is $\dim_{\mathbb{C}} V_{\mathbb{C}} = n$ over $\mathbb{C}$ and $\dim_{\mathbb{R}} V_{\mathbb{C}} = 2n$ over $\mathbb{R}$. It can be noticed that

$$V_{\mathbb{C}} = \mathbb{C} \otimes V. \tag{2.31}$$

By the nature of the tensor product, any vector $\mathbf{v} \in V_{\mathbb{C}}$ is written in a unique way as [8]

$$\mathbf{v} = \mathbf{v}_1 \otimes 1 + \mathbf{v}_2 \otimes i, \tag{2.32}$$

such that $\mathbf{v}_1, \mathbf{v}_2 \in V$. However, for simplicity, we can just write $\mathbf{v} = \mathbf{v}_1 + i\mathbf{v}_2$ as shown in the Definition 2.12.

Given a symmetric bilinear form g endowing V, one can define its extension $g_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ as

$$g_{\mathbb{C}}(\mathbf{v}, \mathbf{u}) = g_{\mathbb{C}}(\mathbf{v}_1 + i\mathbf{v}_2, \mathbf{u}_1 + i\mathbf{u}_2) = g(\mathbf{v}_1, \mathbf{u}_1) - g(\mathbf{v}_2, \mathbf{u}_2) + i(g(\mathbf{v}_1, \mathbf{u}_2) + g(\mathbf{v}_2, \mathbf{u}_1)). \tag{2.33}$$

In the first Chapter, the Clifford algebras have been introduced with the definition of **real** Clifford algebras. We are now in a position to present the *complex Clifford algebras* through the next theorem that unravel its structure.

Theorem 2.13. *Let (V, g) a quadratic space over $\mathbb{R}$ and $\mathcal{C}\ell(V, g)$ its associated Clifford real algebra. Consider the complex Clifford algebra $\mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}})$ for the complexified quadratic space $(V_{\mathbb{C}}, g_{\mathbb{C}})$. Hence*

$$\mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) \simeq \mathcal{C}\ell_{\mathbb{C}}(V, g), \tag{2.34}$$

where $\mathcal{C}\ell_{\mathbb{C}}(V, g) = \mathbb{C} \otimes \mathcal{C}\ell(V, g)$ denotes the complexification of $\mathcal{C}\ell(V, g)$.

Proof. Let us first notice that $\mathbb{C} \otimes \mathcal{C}\ell(V, g)$ is an algebra such that for all $a, b \in \mathbb{C}$ and $A, B \in \mathcal{C}\ell(V, g)$ the product is given by

$$(a \otimes A)(b \otimes B) = ab \otimes AB. \tag{2.35}$$

Since $\dim_{\mathbb{R}} \mathcal{C}\ell(V,g) = 2^{\dim V}$, the dimension of $\mathcal{C}\ell_{\mathbb{C}}(V,g)$ over $\mathbb{R}$ is given by $\dim_{\mathbb{R}} \mathcal{C}\ell_{\mathbb{C}}(V,g) = 2\dim_{\mathbb{R}} \mathcal{C}\ell(V,g) = 2 \cdot 2^{\dim V}$. If $\gamma : V \to \mathcal{C}\ell(V,g)$ denotes the Clifford map, one may define a map $\Gamma : V_{\mathbb{C}} \to \mathcal{C}\ell_{\mathbb{C}}(V,g)$ as a linear map on $\mathbb{C}$ such that

$$\Gamma = 1 \otimes \gamma, \tag{2.36}$$

where 1 denotes the identity of $V_{\mathbb{C}}$. Hence, for $a \otimes \mathbf{v} \in \mathbb{C} \otimes V$ we have that

$$\Gamma(a \otimes \mathbf{v}) = a \otimes \gamma(\mathbf{v}). \tag{2.37}$$

We claim that the map Γ is a Clifford map. Indeed,

$$(\Gamma(1 \otimes \mathbf{v}))^2 = (1 \otimes \gamma(\mathbf{v}))(1 \otimes \gamma(\mathbf{v})) = 1 \otimes g(\mathbf{v}, \mathbf{v}). \tag{2.38}$$

On the other hand, since $\mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}})$ with a Clifford map $\gamma_{\mathbb{C}}$ is a universal Clifford algebra for $(V_{\mathbb{C}}, g_{\mathbb{C}})$, by the universal property there exists a homomorphism $\phi : \mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) \to \mathcal{C}\ell_{\mathbb{C}}(V,g)$ for which the following diagram commutes

$$\begin{tikzcd}
V_{\mathbb{C}} \arrow[r, "\gamma_{\mathbb{C}}"] \arrow[d, "\Gamma"'] & \mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) \arrow[dl, dashed, "\phi"] \\
\mathcal{C}\ell_{\mathbb{C}}(V,g) = \mathbb{C} \otimes \mathcal{C}\ell(V,g) &
\end{tikzcd}$$

With respect to the dimension, we have that: $\dim_{\mathbb{C}} \mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) = 2^{\dim V}$ or equivalently $\dim_{\mathbb{R}} \mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) = 2 \cdot 2^{\dim V}$. Since $\dim \mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) = \dim \mathcal{C}\ell_{\mathbb{C}}(V,g)$ we conclude that ϕ is an isomorphism. □

The above result shows us that the complexification of a Clifford algebra is isomorphic to the Clifford algebra of the complexified vector space. It means that the tensor product with respect to $\mathbb{C}$ can come out of the parentheses, that is,

$$\mathcal{C}\ell(\mathbb{C} \otimes V, g_{\mathbb{C}}) \simeq \mathbb{C} \otimes \mathcal{C}\ell(V,g). \tag{2.39}$$

Once the real Clifford algebras structure is known, the complex Clifford algebras are immediately obtained via the complexification. It alludes that in order to describe the Clifford algebras structure it suffices to understand the real ones. The following theorem describes the periodicity of Clifford algebras and it is important to the construction of the classification.

Theorem 2.14. *Let $\mathcal{C}\ell_{p,q}$ be the Clifford algebra associated with the quadratic space $\mathbb{R}^{p,q}$. Then the following isomorphisms hold:*

$$\begin{aligned} &\textbf{(i)} \quad \mathcal{C}\ell_{p+1,q+1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{p,q}; \\ &\textbf{(ii)} \quad \mathcal{C}\ell_{q+2,p} \simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{p,q}; \\ &\textbf{(iii)} \quad \mathcal{C}\ell_{q,p+2} \simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{p,q}. \end{aligned} \tag{2.40}$$

where either $p,q > 0$ and $\otimes$ denotes the usual tensor product.

Proof. Let U be a 2-dimensional space, with orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ endowed with a symmetric bilinear form g_U such that for $\mathbf{u} = u^1\mathbf{f}_1 + u^2\mathbf{f}_2 \in U$, one may write

$$g_U = \lambda_1(u^1)^2 + \lambda_2(u^2)^2, \quad \lambda_1, \lambda_2 = \pm 1. \tag{2.41}$$

That is,

$$g_U = \begin{cases} (u^1)^2 + (u^2)^2, & \text{if } U = \mathbb{R}^{2,0}, \\ (u^1)^2 - (u^2)^2, & \text{if } U = \mathbb{R}^{1,1}, \\ -(u^1)^2 - (u^2)^2, & \text{if } U = \mathbb{R}^{0,2}. \end{cases} \tag{2.42}$$

With respect to the n-dimensional quadratic space $\mathbb{R}^{p,q}$, $n = p+q$, with an orthonormal basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, for $\mathbf{v} = v^i\mathbf{e}_i \in \mathbb{R}^{p,q}$ we have that

$$g(\mathbf{v},\mathbf{v}) = (v^1)^2 + \cdots + (v^p)^2 - (v^{p+1})^2 - \cdots - (v^{p+q})^2. \tag{2.43}$$

A linear map $\Gamma : \mathbb{R}^{p,q} \oplus U \to \mathcal{C}\ell(U, g_U) \otimes \mathcal{C}\ell_{p,q}$ is defined for all $\mathbf{u} \in U$ and $\mathbf{v} \in \mathbb{R}^{p,q}$ as

$$\Gamma(\mathbf{v} + \mathbf{u}) = \mathbf{f}_1\mathbf{f}_2 \otimes \mathbf{v} + \mathbf{u} \otimes 1 \tag{2.44}$$

where $\mathbf{f}_1 = \rho(\mathbf{f}_1)$, $\mathbf{f}_2 = \rho(\mathbf{f}_2)$, $\mathbf{u} = \rho(\mathbf{u})$, $\mathbf{v} = \gamma(\mathbf{v})$ and $\rho : U \to \mathcal{C}\ell(U, g_U)$, $\gamma : \mathbb{R}^{p,q} \to \mathcal{C}\ell_{p,q}$ are the Clifford mappings. We claim that Γ is also a Clifford map. In fact,

$$\begin{aligned} (\Gamma(\mathbf{v} + \mathbf{u}))^2 &= (\mathbf{f}_1\mathbf{f}_2 \otimes \mathbf{v} + \mathbf{u} \otimes 1)(\mathbf{f}_1\mathbf{f}_2 \otimes \mathbf{v} + \mathbf{u} \otimes 1) \\ &= (\mathbf{f}_1\mathbf{f}_2)^2 \otimes \mathbf{v}^2 + (\mathbf{u}\mathbf{f}_1\mathbf{f}_2 + \mathbf{f}_1\mathbf{f}_2\mathbf{u}) \otimes \mathbf{v} + \mathbf{u}^2 \otimes 1. \end{aligned} \tag{2.45}$$

Recalling that $\{\mathbf{f}_1, \mathbf{f}_2\}$ is orthonormal, it follows that

$$\mathbf{f}_1\mathbf{f}_2 = \mathbf{f}_1 \wedge \mathbf{f}_2 = -\mathbf{f}_2\mathbf{f}_1. \tag{2.46}$$

It then yields

$$\begin{aligned}\mathbf{u}\mathbf{f}_1\mathbf{f}_2+\mathbf{f}_1\mathbf{f}_2\mathbf{u} &= \mathbf{u}(\mathbf{f}_1\wedge\mathbf{f}_2)+(\mathbf{f}_1\wedge\mathbf{f}_2)\mathbf{u} \\ &= 2\mathbf{u}\wedge\mathbf{f}_1\wedge\mathbf{f}_2 \\ &= 2(u^1\mathbf{f}_1+u^2\mathbf{f}_2)\wedge\mathbf{f}_1\wedge\mathbf{f}_2 = 0.\end{aligned} \tag{2.47}$$

We also have that

$$(\mathbf{f}_1\mathbf{f}_2)^2 = \mathbf{f}_1\mathbf{f}_2\mathbf{f}_1\mathbf{f}_2 = -(\mathbf{f}_1)^2(\mathbf{f}_2)^2 = -\lambda_1\lambda_2. \tag{2.48}$$

Therefore, for $(\Gamma(\mathbf{v}+\mathbf{u}))^2$ in Eq. (2.45), it reads

$$\begin{aligned}(\Gamma(\mathbf{v}+\mathbf{u}))^2 &= -\lambda_1\lambda_2\otimes g(\mathbf{v},\mathbf{v})+g_U(\mathbf{u},\mathbf{u})\otimes 1 \\ &= [\lambda_1(u^1)^2+\lambda_2(u^2)^2-\lambda_1\lambda_2((v^1)^2 \\ &\quad +\cdots+(v^p)^2-(v^{p+1})^2-\cdots-(v^{p+q})^2)]1\otimes 1\end{aligned} \tag{2.49}$$

Then, the map Γ is a Clifford map with $\Gamma:\mathbb{R}^{p,q}\oplus U\to \mathcal{C}\ell(W,g_W)$ where W is a $(n+2)$-dimensional space endowed with a symmetric bilinear functional g_W given by

$$g_W = \lambda_1(u^1)^2+\lambda_2(u^2)^2-\lambda_1\lambda_2((v^1)^2+\cdots+(v^p)^2-(v^{p+1})^2-\cdots-(v^{p+q})^2) \tag{2.50}$$

where $\mathbf{w}=\mathbf{u}+\mathbf{v}=u^1\mathbf{f}_1+u^2\mathbf{f}_2+v^i\mathbf{e}_i$. Hence, three possibilities arise:

(i) If $U=\mathbb{R}^{1,1}$ then $W=\mathbb{R}^{p+1,q+1}$, since

$$g_W = (u^1)^2-(u^2)^2+((v^1)^2+\cdots+(v^p)^2-(v^{p+1})^2-\cdots-(v^{p+q})^2). \tag{2.51}$$

(ii) If $U=\mathbb{R}^{2,0}$ then $W=\mathbb{R}^{q+2,p}$, since

$$g_W = (u^1)^2+(u^2)^2-((v^1)^2+\cdots+(v^p)^2-(v^{p+1})^2-\cdots-(v^{p+q})^2). \tag{2.52}$$

(iii) If $U=\mathbb{R}^{0,2}$ then $W=\mathbb{R}^{q,p+2}$, since

$$g_W = -(u^1)^2-(u^2)^2-((v^1)^2+\cdots+(v^p)^2-(v^{p+1})^2-\cdots-(v^{p+q})^2). \tag{2.53}$$

In addition, the isomorphism follows from the Clifford algebra universality

$$\begin{tikzcd}
\mathbb{R}^{p,q}\oplus U \arrow[r] \arrow[d] & \mathcal{C}\ell(W,g_W) \arrow[dl, dashed, "\simeq"] \\
\mathcal{C}\ell(U,g_U)\otimes\mathcal{C}\ell_{p,q} &
\end{tikzcd}$$

That is,

$$\begin{tikzcd}
\mathbb{R}^{p+1,q+1} \arrow[r] \arrow[d] & \mathcal{C}\ell_{p+1,q+1} \arrow[dl, dashed, "\simeq"] \\
\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{p,q} &
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbb{R}^{q+2,p} \arrow[r] \arrow[d] & \mathcal{C}\ell_{q+2,p} \arrow[dl, dashed, "\simeq"] \\
\mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{p,q} &
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbb{R}^{q,p+2} \arrow[r] \arrow[d] & \mathcal{C}\ell_{q,p+2} \arrow[dl, dashed, "\simeq"] \\
\mathcal{C}\ell_{0,2}\otimes\mathcal{C}\ell_{p,q} &
\end{tikzcd}$$

which proves the Theorem. □

By combining the above isomorphisms several others can be obtained. For instance, notice that the repeated use of the relation **(i)** in the Theorem 2.14 yields

$$\mathcal{C}\ell_{p,p} \simeq \overbrace{\mathcal{C}\ell_{1,1}\otimes\cdots\otimes\mathcal{C}\ell_{1,1}}^{p\text{ factors}} = \otimes^p\,\mathcal{C}\ell_{1,1}. \tag{2.54}$$

By the same relation **(i)** in the Theorem 2.14, for $p < q$ it follows that

$$\begin{aligned}
\mathcal{C}\ell_{p,q} &\simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{p-1,q-1}\\
&\simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{p-2,q-2}\\
&\;\vdots\\
&\simeq \otimes^p\,\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{0,q-p}\\
&\simeq \mathcal{C}\ell_{p,p}\otimes\mathcal{C}\ell_{0,q-p}.
\end{aligned} \tag{2.55}$$

Therefore, we have the following relations

$$\begin{aligned}
\mathcal{C}\ell_{p,q} &\simeq \mathcal{C}\ell_{p,p}\otimes\mathcal{C}\ell_{0,q-p} \quad (p<q),\\
\mathcal{C}\ell_{p,q} &\simeq \mathcal{C}\ell_{q,q}\otimes\mathcal{C}\ell_{p-q,0} \quad (p>q).
\end{aligned} \tag{2.56}$$

In particular, by the relation **(iii)** in Theorem 2.14 it is straightforward that

$$\mathcal{C}\ell_{0,4} \simeq \mathcal{C}\ell_{0,2}\otimes\mathcal{C}\ell_{2,0}. \tag{2.57}$$

and that

$$\mathcal{C}\ell_{2,2} \simeq \mathcal{C}\ell_{0,2}\otimes\mathcal{C}\ell_{2,0} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,1}. \tag{2.58}$$

Therefore, by the relations **(ii)** and **(iii)** in the Theorem 2.14 and the Eq. (2.57) it follows that

$$\begin{aligned}\mathcal{C}\ell_{p,q+4} = \mathcal{C}\ell_{p,(q+2)+2} &\simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{q+2,p}\\ &\simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{p,q}\\ &\simeq \mathcal{C}\ell_{0,4} \otimes \mathcal{C}\ell_{p,q},\end{aligned} \tag{2.59}$$

which implies the following relation

$$\mathcal{C}\ell_{0,8} \simeq \mathcal{C}\ell_{0,4} \otimes \mathcal{C}\ell_{0,4}. \tag{2.60}$$

It also follows that

$$\begin{aligned}\mathcal{C}\ell_{p,q+8} = \mathcal{C}\ell_{p,(q+4)+4} &\simeq \mathcal{C}\ell_{0,4} \otimes \mathcal{C}\ell_{p,q+4}\\ &\simeq \mathcal{C}\ell_{0,4} \otimes \mathcal{C}\ell_{0,4} \otimes \mathcal{C}\ell_{p,q}\\ &\simeq \mathcal{C}\ell_{0,8} \otimes \mathcal{C}\ell_{p,q}.\end{aligned} \tag{2.61}$$

Another isomorphism that does not follow from the Theorem 2.14 that has outstanding importance is given in the next lemma.

Lemma 2.15. $\mathcal{C}\ell_{2,0} \simeq \mathcal{C}\ell_{1,1}$.

Proof. Each $A \in \mathcal{C}\ell_{2,0}$ and $B \in \mathcal{C}\ell_{1,1}$ has the form

$$\begin{aligned}A &= a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_{12}\mathbf{e}_1\mathbf{e}_2,\\ B &= b_0 + b_1\mathbf{f}_1 + b_2\mathbf{f}_2 + b_{12}\mathbf{f}_1\mathbf{f}_2.\end{aligned} \tag{2.62}$$

Such that those basis elements satisfy the following relations

$$\begin{aligned}(\mathbf{e}_1)^2 = 1,\ (\mathbf{e}_2)^2 = 1;\\ (\mathbf{f}_1)^2 = 1,\ (\mathbf{f}_1)^2 = -1.\end{aligned} \tag{2.63}$$

Therefore, one can define a linear map $\phi : \mathcal{C}\ell_{2,0} \to \mathcal{C}\ell_{1,1}$ by setting

$$\phi(1) = 1, \quad \phi(\mathbf{e}_1) = \mathbf{f}_1, \quad \phi(\mathbf{e}_2) = \mathbf{f}_1, \quad \phi(\mathbf{e}_1\mathbf{e}_2) = \mathbf{f}_2, \tag{2.64}$$

which is an algebra isomorphism. □

By the relations **(i)** and **(ii)** in the Theorem 2.14 and the above Lemma 2.15, another isomorphism can be derived:

$$\begin{aligned}\mathcal{C}\ell_{p+1} &\simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{p,q-1}\\ &\simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{p,q-1}\\ &\simeq \mathcal{C}\ell_{q+1,p}.\end{aligned} \tag{2.65}$$

Such results indicate that by knowing the following low-dimensional Clifford algebras

$$\mathcal{C}\ell_{1,0}, \quad \mathcal{C}\ell_{0,1}, \quad \mathcal{C}\ell_{0,2}, \quad \mathcal{C}\ell_{1,1} \simeq \mathcal{C}\ell_{2,0} \tag{2.66}$$

then we know all of them in arbitrary finite dimensions since any other Clifford algebras can be constructed using the isomorphisms that we presented. Those isomorphisms introduce a method for the classification of Clifford algebras that shall be regarded later on. It is worth pointing out that the Theorem 2.14 can be generalised as follows

Corollary 2.16. *Let $\mathcal{C}\ell_{p,q}$ be a Clifford algebra associated with the quadratic space $\mathbb{R}^{p,q}$ and let $k > 0$. Then the following isomorphisms hold:*

$$\begin{aligned} &\textbf{(i)} \quad \mathcal{C}\ell_{p+k,q+k} \simeq \mathcal{C}\ell_{k,k} \otimes \mathcal{C}\ell_{p,q};\\ &\textbf{(ii)} \quad \mathcal{C}\ell_{q+2k,p} \simeq \mathcal{C}\ell_{2k,0} \otimes \mathcal{C}\ell_{p,q};\\ &\textbf{(iii)} \quad \mathcal{C}\ell_{q,p+2k} \simeq \mathcal{C}\ell_{0,2k} \otimes \mathcal{C}\ell_{p,q}. \end{aligned} \tag{2.67}$$

where either $p, q > 0$ and $\otimes$ denotes the usual tensor product.

Proof. Let U be a $2k$-dimensional space such that $U = \mathbb{R}^{k,k}, U = \mathbb{R}^{0,2k}, U = \mathbb{R}^{2k,0}$. Therefore, the same steps shown in the proof of the Theorem 2.14 provide the desired isomorphisms. □

Finally, we also have the following result about isomorphisms regarding the even subalgebra structure:

Theorem 2.17. *Let $\mathcal{C}\ell_{p,q}$ a Clifford algebra associated with the quadratic space $\mathbb{R}^{p,q}$ and $\mathcal{C}\ell^{+}_{p,q}$ its even subalgebra. Then the following isomorphisms hold:*

$$\mathcal{C}\ell^{+}_{p,q} \simeq \mathcal{C}\ell_{q,p-1} \simeq \mathcal{C}\ell_{p,q-1} \simeq \mathcal{C}\ell^{+}_{q,p}. \tag{2.68}$$

Proof. Let $\{\mathbf{e}_i, \mathbf{f}_k\}$ with $i \in \{1,..,p\}$ and $k \in \{1,..,q\}$ an orthonormal basis of the vector space V such that $\mathcal{C}\ell_{p,q}$ is generated by 1 and $\{\mathbf{e}_i, \mathbf{f}_k\}$, for $i, j \in \{1,..,p\}$ and $k, l \in \{1,..,q\}$ such that

$$\begin{aligned} &(\mathbf{e}_i)^2 = 1, \quad (\mathbf{f}_k)^2 = -1,\\ &\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0 \ \ (i \neq j),\\ &\mathbf{f}_k\mathbf{f}_l + \mathbf{f}_l\mathbf{f}_k = 0 \ \ (k \neq l),\\ &\mathbf{e}_i\mathbf{f}_k + \mathbf{f}_k\mathbf{e}_i = 0. \end{aligned} \tag{2.69}$$

Then the vector space $\bigwedge^2(\mathbb{R}^{p,q})$ consists of the elements $\{\mathbf{e}_i\mathbf{e}_j\ (i \neq j), \mathbf{f}_k\mathbf{f}_l\ (k \neq l), \mathbf{e}_i\mathbf{f}_l\}$. However, not all of those quantities generate the even subalgebra $\mathcal{C}\ell_{p,q}^+$. Actually, there is a redundancy. For example, all the bivectors $\{\mathbf{f}_k\mathbf{f}_l\ (k \neq l)\}$ may be written in terms of the bivectors of type $\{\mathbf{e}_i\mathbf{f}_l\}$, since

$$(\mathbf{e}_i\mathbf{f}_k)(\mathbf{e}_i\mathbf{f}_l) = -(\mathbf{e}_i)^2\mathbf{f}_k\mathbf{f}_l = -\mathbf{f}_k\mathbf{f}_l \ \ (k \neq l). \tag{2.70}$$

Choosing an arbitrary vector, for instance, $\mathbf{e}_1$, it follows that the set $\{\mathbf{e}_1, \mathbf{e}_m, \mathbf{e}_1, \mathbf{f}_k\}$ with $m \in \{2, ..., p\}$ and $k \in 1, .., q$ generates the space $\bigwedge^2(\mathbb{R}^{p,q})$ and then it generates the even subalgebra $\mathcal{C}\ell_{p,q}^+$. Writing such generators of $\mathcal{C}\ell_{p,q}^+$ as $\xi_a = \mathbf{e}_1\mathbf{e}_{a+1}$ for $a \in \{1, ..., p-1\}$ and $\zeta_b = \mathbf{e}_1\mathbf{f}_b$ for $b \in \{1, ..., q\}$ it yields

$$\begin{aligned}
&(\xi_a)^2 = -(\mathbf{e}_1)^2(\mathbf{e}_{a+1})^2 = -1\\
&(\zeta_b)^2 = -(\mathbf{e}_1)^2(\mathbf{f}_b)^2 = 1\\
&\xi_a\zeta_b + \zeta_b\xi_a = 0\\
&\xi_a\xi_c + \xi_c\xi_a = 0 \ \ (a \neq c)\\
&\zeta_b\zeta_d + \zeta_d\zeta_b = 0 \ \ (b \neq d)
\end{aligned} \tag{2.71}$$

Therefore, the quantities $\{\zeta_b, \xi_a\}$ for $b \in \{1, ..., q\}$ and $a \in \{1, ..., p-1\}$ are the generators of a Clifford algebra associated with the quadratic space $\mathbb{R}^{q,p-1}$, that is, $\mathcal{C}\ell_{p,q}^+ \simeq \mathcal{C}\ell_{q,p-1}$. The other above isomorphisms naturally follow from the isomorphism written in the Eq. (2.65). □

3 CLASSIFICATION OF CLIFFORD ALGEBRAS

Finally, in this Chapter we will explore the periodicity of finite dimensional Clifford algebras and classify all of them. The prominence of representation theory will also be concerned, since Clifford algebra is naturally related to matrix algebra and their classification is based on it. The representation theory in algebra is very important by at least the following reasons: a priori, the Clifford algebras seem to be very abstract, however, from their representation by matrix algebra we can see explicit some relevant and concrete properties and behaviour of those algebraic structures. The calculations involving matrix algebra is oftentimes easier and simple rather than the calculations on the Clifford algebra abstract structure. In addition, the matrix algebra dialogue with an abundance of areas and the natural relation of them with Clifford algebras, makes the Clifford algebra rich in applications. Moreover, based on the periodicity of the Clifford algebra, we will work on the representation of the low-dimensional Clifford algebras to develop the classification of all of them as our main goal.

Definition 3.1. *Let $\mathcal{A}$ be a real algebra and V a vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. A linear map $\rho : \mathcal{A} \to End_{\mathbb{K}}(V)$ satisfying $\rho(1_{\mathcal{A}}) = 1_V$ and $\rho(ab) = \rho(a)\rho(b)$, for all $a, b \in \mathcal{A}$, is called a* $\mathbb{K}$-***representation*** *of $\mathcal{A}$.*

- Such vector space V is called the *representation space* or *carrier space* of $\mathcal{A}$.
- Two representations $\rho_1 : \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(V_1)$ and $\rho_2 : \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(V_2)$ are *equivalent* if there exists a $\mathbb{K}$-isomorphism $\phi : V_1 \to V_2$ satisfying $\rho_2(a) = \phi \circ \rho_1(a) \circ \phi^{-1}$, for all $a \in \mathcal{A}$.
- A representation is said to be *faithful* if $\ker \rho = \{0\}$.
- A representation is *irreducible* or *simple* if the only invariant subspaces of $\rho(a)$, for all $a \in \mathcal{A}$ are V and $\{0\}$. It is said to be *reducible* or *semisimple* if $V = V_1 \oplus V_2$ where V_1 and V_2 are invariant subspaces under the action of $\rho(a)$, for all $a \in \mathcal{A}$.

Example 3.2 ▹ Representations for $\mathbb{C}$.

Consider the algebra of complex numbers $\mathbb{C}$. As an algebra over $\mathbb{C}$ it has two representations $\rho(a+ib) = a+ib$ and $\bar{\rho}(a+ib) = a-ib$. Since there does not exists any linear map $\phi_z : \mathbb{C} \to \mathbb{C}; (a+ib) \mapsto \phi_z(a+ib) = z(a+ib)$, where $z = x+iy$, s.t., $\bar{\rho}(a+ib) = z\rho(a+ib)z^{-1}$, these two $\mathbb{C}$-representations are not equivalent. Although, every $\mathbb{C}$-representation (and also every $\mathbb{H}$-representation) is a $\mathbb{R}$-representation. One can define two real representations $\sigma : \mathbb{C} \to \mathcal{M}(2,\mathbb{R})$ and $\bar{\sigma} : \mathbb{C} \to \mathcal{M}(2,\mathbb{R})$ as

$$\sigma(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \bar{\sigma}(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad . \tag{3.1}$$

Such representations are equivalent, namely, there exists an isomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\bar{\sigma}(a+ib) = \phi\sigma(a+ib)\phi^{-1}$. For instance,

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \phi^{-1}. \tag{3.2}$$

Those $\mathbb{R}$-representations are irreducible. An example of a reducible $\mathbb{R}$-representation is provided by

$$\xi(a+ib) = \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}. \triangleleft \tag{3.3}$$

Example 3.3 ▹ The representation of $\mathcal{C}\ell_3$ by $\mathcal{M}(2,\mathbb{C})$.

Let us recall the Clifford algebras $\mathcal{M}(2,\mathbb{C})$ generated by the Pauli matrices $\{1,\sigma_1,\sigma_2,\sigma_3\}$ described in the Example 1.8 and the universal Clifford algebra $\mathcal{C}\ell_3$ described in the Example 1.9. By setting the following identification in the generators:

$$\mathbf{e}_1 \leftrightarrow \sigma_1, \quad \mathbf{e}_2 \leftrightarrow \sigma_2, \quad \mathbf{e}_3 \leftrightarrow \sigma_3, \tag{3.4}$$

we have the following correspondence of the basis elements of those Clifford algebras:

$\mathcal{M}(2,\mathbb{C})$	$\mathcal{C}\ell_3$
I	1
$\sigma_1,\ \sigma_2,\ \sigma_3$	$\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$
$\sigma_1\sigma_2,\ \sigma_1\sigma_3,\ \sigma_2\sigma_3$	$\mathbf{e}_1\mathbf{e}_2,\ \mathbf{e}_1\mathbf{e}_3,\ \mathbf{e}_2\mathbf{e}_3$
$\sigma_1\sigma_2\sigma_3$	$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$

Therefore, by that identification, an arbitrary element $A \in \mathcal{C}\ell_3$

$$A = a + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{12}\mathbf{e}_1\mathbf{e}_2 + a_{13}\mathbf{e}_1\mathbf{e}_3 + a_{23}\mathbf{e}_2\mathbf{e}_3 + a_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \tag{3.5}$$

can be written in terms of $\mathcal{M}(2,\mathbb{C})$ as:

$$A \leftrightarrow \begin{pmatrix} (a+a_3)+i(a_{12}+a_{123}) & (a_1-a_{13})-i(a_2-a_{23}) \\ (a_1+a_{13})+i(a_2+a_{23}) & (a-a_3)-i(a_{12}-a_{123}) \end{pmatrix} = \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}. \tag{3.6}$$

The operations of grade involution, reversion and conjugation defined in the Section 2.2 in $\mathcal{C}\ell_3$ are given respectively by:

$$\begin{aligned} \hat{A} &= A_0 - A_1 + A_2 - A_3, \\ \widetilde{A} &= A_0 + A_1 - A_2 - A_3, \\ \overline{A} &= A_0 - A_1 - A_2 + A_3. \end{aligned} \tag{3.7}$$

Right, in terms of $\mathcal{M}(2,\mathbb{C})$ they are represented, respectively, by the following matrices:

$$\begin{aligned} \hat{A} &\leftrightarrow \begin{pmatrix} (a-a_3)-i(a_{123}-a_{12}) & (-a_{13}-a_1)+i(a_2+a_{23}) \\ (a_{13}-a_1)+i(a_{23}-a_2) & (a+a_3)-i(a_{12}+a_{123}) \end{pmatrix} = \begin{pmatrix} z_4^* & -z_2^* \\ -z_3^* & z_1^* \end{pmatrix}, \\ \widetilde{A} &\leftrightarrow \begin{pmatrix} (a+a_3)-i(a_{12}+a_{123}) & (a_1+a_{13})-i(a_2+a_{23}) \\ (a_1-a_{13})+i(a_2-a_{23}) & (a-a_3)+i(a_{12}-a_{123}) \end{pmatrix} = \begin{pmatrix} z_1^* & z_3^* \\ z_2^* & z_4^* \end{pmatrix}, \\ \overline{A} &\leftrightarrow \begin{pmatrix} (a-a_3)-i(a_{12}-a_{123}) & (a_{13}-a_1)+i(a_2-a_{23}) \\ (-a_1-a_{13})-i(a_2+a_{23}) & (a+a_3)+i(a_{12}+a_{123}) \end{pmatrix} = \begin{pmatrix} z_4 & -z_3 \\ -z_2 & z_1 \end{pmatrix}. \end{aligned} \tag{3.8}$$

In addition, the elements of the odd part $A_- \in \mathcal{C}\ell_3^-$, the even subalgebra $A_+ \in \mathcal{C}\ell_3^+$, and the centre $A_{cen} \in Cen(\mathcal{C}\ell_3)$ in terms of the algebra $\mathcal{M}(2,\mathbb{C})$ they are given by:

$$\begin{aligned} A_- = A_1 + A_3 &\leftrightarrow \begin{pmatrix} a_3 + ia_{123} & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 + ia_{123} \end{pmatrix} = \begin{pmatrix} w_1 & w_2^* \\ w_2 & -w_1^* \end{pmatrix}, \\ A_+ = A_0 + A_2 &\leftrightarrow \begin{pmatrix} a + ia_{12} & -a_{13} + ia_{23} \\ a_{13} + ia_{23} & a - ia_{12} \end{pmatrix} = \begin{pmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{pmatrix}, \\ A_{cen} = A_0 + A_3 &\leftrightarrow \begin{pmatrix} a + ia_{123} & 0 \\ 0 & a + ia_{123} \end{pmatrix} = \begin{pmatrix} y_1 & 0 \\ 0 & y_1 \end{pmatrix}. \end{aligned} \tag{3.9}$$ ◁

Example 3.4 ▷ The representation of $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ by $\mathcal{M}(4,\mathbb{C})$.

Let us consider now the complex Clifford algebra $\mathcal{C}\ell(\mathbb{C} \otimes V, g_{\mathbb{C}}) = \mathbb{C} \otimes \mathcal{C}\ell(V,g)$, this algebra is isomorphic to the matrix algebra $\mathcal{M}(4,\mathbb{C})$ generated by the Dirac matrices

$\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ presented in the Introduction of this work in the Eq. (0.6). We identify the generators of $\mathcal{C}\ell_{1,3}$ with the Dirac matrices

$$\mathbf{e}_0 \leftrightarrow \gamma_0, \quad \mathbf{e}_1 \leftrightarrow \gamma_1, \quad \mathbf{e}_2 \leftrightarrow \gamma_2, \quad \mathbf{e}_3 \leftrightarrow \gamma_3. \tag{3.10}$$

Therefore, we can write an arbitrary element $C \in \mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ in terms of $\mathcal{M}(4,\mathbb{C})$. This element has the form:

$$\begin{aligned}
&C = A + iB \quad A, B \in \mathcal{C}\ell_{1,3}\\
&C = a + a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{01}\mathbf{e}_0\mathbf{e}_1 + a_{02}\mathbf{e}_0\mathbf{e}_2 + a_{03}\mathbf{e}_0\mathbf{e}_3 + a_{12}\mathbf{e}_1\mathbf{e}_2\\
&\quad + a_{13}\mathbf{e}_1\mathbf{e}_3 + a_{23}\mathbf{e}_2\mathbf{e}_3 + a_{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + a_{013}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3 + a_{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + a_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\\
&\quad + a_{0123}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + i(b + b_0\mathbf{e}_0 + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_{01}\mathbf{e}_0\mathbf{e}_1 + b_{02}\mathbf{e}_0\mathbf{e}_2\\
&\quad + b_{03}\mathbf{e}_0\mathbf{e}_3 + b_{12}\mathbf{e}_1\mathbf{e}_2 + b_{13}\mathbf{e}_1\mathbf{e}_3 + b_{23}\mathbf{e}_2\mathbf{e}_3 + b_{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + b_{013}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3\\
&\quad + b_{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + b_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + b_{0123}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3).
\end{aligned} \tag{3.11}$$

In terms of $\mathcal{M}(4,\mathbb{C})$ we have that:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i\\ 0 & 0 & -i & 0\\ 0 & -i & 0 & 0\\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \gamma_{01} = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}, \gamma_{02} = \begin{pmatrix} 0 & 0 & 0 & i\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_{03} = \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\gamma_{12} = \begin{pmatrix} -i & 0 & 0 & 0\\ 0 & i & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & 0 & 0 & i \end{pmatrix}, \quad \gamma_{13} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{pmatrix}, \gamma_{23} = \begin{pmatrix} 0 & -i & 0 & 0\\ -i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & -i & 0 \end{pmatrix}, \quad \gamma_{012} = \begin{pmatrix} -i & 0 & 0 & 0\\ 0 & i & 0 & 0\\ 0 & 0 & i & 0\\ 0 & 0 & 0 & -i \end{pmatrix},$$

$$\gamma_{013} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \gamma_{023} = \begin{pmatrix} 0 & -i & 0 & 0\\ -i & 0 & 0 & 0\\ 0 & 0 & 0 & i\\ 0 & 0 & i & 0 \end{pmatrix}, \gamma_{123} = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma_{0123} = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Applying the correspondence, we have that:

$$
\begin{aligned}
&C_0 = aI + ibI,\\
&C_1 = a_0\gamma_0 + a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 + i(b_0\gamma_0 + b_1\gamma_1 + b_2\gamma_2 + b_3\gamma_3),\\
&C_2 = a_{01}\gamma_{01} + a_{02}\gamma_{02} + a_{03}\gamma_{03} + a_{12}\gamma_{12} + a_{13}\gamma_{13} + a_{23}\gamma_{23} + i(b_{01}\gamma_{01} + b_{02}\gamma_{02} + b_{03}\gamma_{03}\\
&+ b_{12}\gamma_{12} + b_{13}\gamma_{13} + b_{23}\gamma_{23}),\\
&C_3 = a_{012}\gamma_{012} + a_{013}\gamma_{013} + a_{023}\gamma_{023} + a_{123}\gamma_{123} + i(b_{012}\gamma_{012} + b_{013}\gamma_{013} + b_{023}\gamma_{023} + b_{123}\gamma_{123}),\\
&C_4 = a_{0123}\gamma_{0123} + ib_{0123}\gamma_{0123}.
\end{aligned}
\tag{3.12}
$$

Therefore $C = C_0 + C_1 + C_2 + C_3 + C_4 \in \mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ is represented by the matrix:

$$
C = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \in \mathcal{M}(4, \mathbb{C}),
\tag{3.13}
$$

such that each term is given by

$$
\begin{aligned}
m_{11} &= (a + a_0 + b_{12} + b_{012}) + i(-a_{12} - a_{012} + b + b_0),\\
m_{12} &= (a_{13} + a_{013} + b_{23} + b_{023}) + i(-a_{23} - a_{023} + b_{13} + b_{013}),\\
m_{13} &= (-a_3 - a_{03} + a_{123} + a_{0123}) + i(-b_3 - b_{03} + b_{123} + b_{0123}),\\
m_{14} &= (-a_1 - a_{01} - b_2 - b_{02}) + i(a_2 + a_{02} - b_1 - b_{01})\\
m_{21} &= (-a_{13} - a_{013} + b_{23} + b_{023}) + i(-a_{23} - a_{023} - b_{13} - b_{013}),\\
m_{22} &= (a + a_0 - b_{12} - b_{012}) + i(a_{12} + a_{012} + b + b_0),\\
m_{23} &= (-a_1 - a_{01} + b_2 + b_{02}) + i(-a_2 - a_{02} + b_1 - b_{01}),\\
m_{24} &= (a_3 + a_{03} - a_{123} - a_{0123}) + i(+b_3 + b_{03} - b_{123} - b_{0123}),\\
m_{31} &= (a_3 - a_{03} - a_{123} + a_{0123}) + i(b_3 - b_{03} - b_{123} + b_{0123}),\\
m_{32} &= (a_1 - a_{01} + b_2 - b_{02}) + i(-a_2 + a_{02} + b_1 - b_{01}),\\
m_{33} &= (a - a_0 + b_{12} - b_{012}) + i(-a_{12} + a_{012} + b - b_0),\\
m_{34} &= (a_{13} - a_{013} + b_{23} - b_{023}) + i(-a_{23} + a_{023} + b_{13} - b_{013}),\\
m_{41} &= (a_1 - a_{01} - b_2 + b_{02}) + i(a_2 - a_{02} + b_1 - b_{01}),\\
m_{42} &= (-a_3 + a_{03} + a_{123} - a_{0123} + b_{23}) + i(-a_{23} - b_3 + b_{03} + b_{123} - b_{0123}),\\
m_{43} &= (-a_{13} + a_{013} - b_{023}) + i(a_{023} - b_{13} + b_{013}),\\
m_{44} &= (a - a_0 - b_{12} + b_{012}) + i(a_{12} - a_{012} + b - b_0).
\end{aligned}
\tag{3.14}
$$

This example provides explicitly the representation of an arbitrary element in the complex Clifford algebra $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$. ◃

Right, let us recall the important result developed in the previous Chapter which tell us that by knowing four low-dimensional Clifford algeras, namely, $\mathcal{C}\ell_{1,0}$, $\mathcal{C}\ell_{0,1}$, $\mathcal{C}\ell_{0,2}$ and $\mathcal{C}\ell_{1,1} \simeq \mathcal{C}\ell_{2,0}$, then we know the next ones by applying the Theorem 2.14. Our next goal, aiming the classification, will be therefore establish the representation of such low-dimensional Clifford algebras.

3.1. The Clifford Algebra $\mathcal{C}\ell_{0,1}$

A Clifford algebra associated to the quadratic space $\mathbb{R}^{0,1}$ was introduced in Example 1.7. If $\mathbf{e}$ is an unit vector such that $g(\mathbf{e},\mathbf{e}) = -1$, an arbitrary element $\psi \in \mathcal{C}\ell_{0,1}$ is written as

$$\psi = a + b\mathbf{e} \in \mathcal{C}\ell_{0,1} \tag{3.15}$$

where $\mathbf{e}^2 = -1$. This algebra is isomorphic to the complex algebra $\mathbb{C}$, namely, the set of pairs $(a,b) \in \mathbb{R}^2$ endowed with the multiplication given by

$$(a,b)(c,d) = (ac - bd, ad + bc). \tag{3.16}$$

The isomorphism is establish by $\rho : \mathcal{C}\ell_{0,1} \to \mathbb{C}$ such that $\rho(1) = (1,0)$ and $\rho(\mathbf{e}) = (0,1) = i$. Therefore

$$\mathcal{C}\ell_{0,1} \simeq \mathbb{C}. \tag{3.17}$$

3.2. The Clifford Algebra $\mathcal{C}\ell_{1,0}$

Let us consider the quadratic space $\mathbb{R}^{1,0}$. Taking the unit vector $\mathbf{e}$ such that $g(\mathbf{e},\mathbf{e}) = 1$, an arbitrary element ψ of $\mathcal{C}\ell_{1,0}$ reads

$$\psi = a + b\mathbf{e} \in \mathcal{C}\ell_{1,0} \tag{3.18}$$

where now $\mathbf{e}^2 = 1$. The difference between this case and the previous one is that here we have $\mathbf{e}^2 = 1$ instead of $\mathbf{e}^2 = -1$, it yields some drastic distinct consequences. To be more precise, let us consider the set of the pairs of numbers $(a,b) \in \mathbb{R}^2$ with multiplication defined by

$$(a,b)(c,d) = (ac + bd, ad + bc). \tag{3.19}$$

Let us denote this set by $\mathbb{D}$, whose elements have different denominations: double numbers, perplex numbers, duplex or Lorentz numbers [7]. In particular, such set is not a field, but a ring. It is neither a division ring, since $(1,1)(1,-1)=(0,0)$. Despite the above defined multiplication being appropriate to compare $\mathbb{D}$ to $\mathbb{C}$, it is not so suitable for the Clifford algebras classification. Therefore, let us consider the set of the pairs of numbers $(a,b)\in\mathbb{R}^2$ endowed with a product defined by

$$(a,b)*(c,d)=(ac,bd) \tag{3.20}$$

This algebra is the direct sum of real algebras, namely, $\mathbb{R}\oplus\mathbb{R}$. It is isomorphic to the 2×2 diagonal matrices. Such isomorphism is given by

$$\phi(a,b)=\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{3.21}$$

The algebra $\mathbb{R}\oplus\mathbb{R}$ is also isomorphic to $\mathbb{D}$. The isomorphism $\phi:\mathbb{D}\to\mathbb{R}\oplus\mathbb{R}$ reads

$$\varphi(a,b)=(a+b,a-b). \tag{3.22}$$

In fact, let $(a,b),(c,d)\in\mathbb{D}$

$$\begin{aligned}\varphi((a,b)(c,d)) &= \varphi(ac+bd,ad+bc)\\ &= (ac+bd+ad+bc,ac+bd-ad-bc)\\ &= ((a+b)(c+d),(a-b)(c-d))\\ &= \varphi(a,b)*\varphi(c,d).\end{aligned} \tag{3.23}$$

On the other hand, $\mathcal{C}\ell_{1,0}$ is isomorphic to $\mathbb{D}$ by the identification $1\leftrightarrow(1,0)$ and $\mathbf{e}\leftrightarrow(0,1)$. Hence,

$$\mathcal{C}\ell_{1,0}\simeq\mathbb{R}\oplus\mathbb{R}. \tag{3.24}$$

3.3. The Clifford Algebra $\mathcal{C}\ell_{0,2}$

Let us consider now the quadratic space $\mathbb{R}^{0,2}$ and an orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2\}$, it follows that

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=-1, \qquad g(\mathbf{e}_1,\mathbf{e}_2)=g(\mathbf{e}_2,\mathbf{e}_1)=0. \tag{3.25}$$

An arbitrary element ψ of $\mathcal{C}\ell_{0,2}$ is written as

$$\psi=a+b\mathbf{e}_1+c\mathbf{e}_2+d\mathbf{e}_1\mathbf{e}_2\in\mathcal{C}\ell_{0,2}, \tag{3.26}$$

such that $a, b, c, d \in \mathbb{R}$ and

$$(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = -1, \quad \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0, \quad (\mathbf{e}_1\mathbf{e}_2)^2 = -1. \tag{3.27}$$

The Clifford algebra $\mathcal{C}\ell_{0,2}$ is isomorphic to the quaternion algebra $\mathbb{H}$ through the identification [7]

$$1 \leftrightarrow 1, \quad \mathbf{e}_1 \leftrightarrow i, \quad \mathbf{e}_2 \leftrightarrow j, \quad \mathbf{e}_1\mathbf{e}_2 \leftrightarrow k, \tag{3.28}$$

$\mathcal{C}\ell_{0,2}$	$\mathbb{H}$
$\mathbf{e}_1$	i
$\mathbf{e}_2$	j
$\mathbf{e}_1\mathbf{e}_2$	k

where i, j, k are the quaternion units

$$\begin{aligned} i^2 = j^2 = k^2 = ijk = -1, \\ jk = -kj = i, \\ ki = -ik = j, \\ ij = -ji = k. \end{aligned} \tag{3.29}$$

Therefore,

$$\mathcal{C}\ell_{0,2} \simeq \mathbb{H}. \tag{3.30}$$

3.4. The Clifford Algebra $\mathcal{C}\ell_{2,0} \simeq \mathcal{C}\ell_{1,1}$

The Clifford algebras associated to the quadratic spaces $\mathbb{R}^{2,0}$ and $\mathbb{R}^{1,1}$ were shown to be isomorphic in Lemma 2.15. Hence, it suffices to consider just one of those spaces. For instance, let us consider $\mathbb{R}^{2,0}$, it holds for an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ that

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_1, \mathbf{e}_2) = g(\mathbf{e}_2, \mathbf{e}_1) = 0. \tag{3.31}$$

One may write an arbitrary element ψ of $\mathcal{C}\ell_{2,0}$ as

$$\psi = a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_1\mathbf{e}_2 \in \mathcal{C}\ell_{2,0}, \tag{3.32}$$

where $a, b, c, d \in \mathbb{R}$ and

$$(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = 1, \quad \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0, \quad (\mathbf{e}_1\mathbf{e}_2)^2 = -1. \tag{3.33}$$

Let $\mathcal{M}(2,\mathbb{R})$ be the set of the real 2×2 matrices. The generator of such algebra is the following set

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}. \tag{3.34}$$

Furthermore,

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^2, \quad \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}^2=-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{3.35}$$

Comparing the relations in the Eqs. (3.34) and (3.35), an isomorphism between those algebras can be constructed by the identification

$$1\leftrightarrow\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \mathbf{e}_1\leftrightarrow\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \mathbf{e}_2\leftrightarrow\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \mathbf{e}_1\mathbf{e}_2\leftrightarrow\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{3.36}$$

Hence,

$$\mathcal{C}\ell_{2,0}\simeq\mathcal{C}\ell_{1,1}\simeq\mathcal{M}(2,\mathbb{R}). \tag{3.37}$$

3.5. Clifford Algebras Classification

Once the isomorphisms

$$\begin{aligned} &\textbf{(i)} && \mathcal{C}\ell_{0,1}\simeq\mathbb{C},\\ &\textbf{(ii)} && \mathcal{C}\ell_{1,0}\simeq\mathbb{R}\oplus\mathbb{R},\\ &\textbf{(iii)} && \mathcal{C}\ell_{0,2}\simeq\mathbb{H},\\ &\textbf{(iv)} && \mathcal{C}\ell_{2,0}\simeq\mathcal{C}\ell_{1,1}\simeq\mathcal{M}(2,\mathbb{R}) \end{aligned} \tag{3.38}$$

has been established, by using the isomorphisms presented in the previous Chapter, the classification of arbitrary Clifford algebras can proceed. For instance, by using Eq. (2.54) and the following relation

$$\mathcal{M}(m,\mathbb{R})\otimes\mathcal{M}(n,\mathbb{R})\simeq\mathcal{M}(mn,\mathbb{R}) \tag{3.39}$$

it follows that

$$\mathcal{C}\ell_{p,p}\simeq\otimes^p\,\mathcal{C}\ell_{1,1}\simeq\otimes^p\,\mathcal{M}(2,\mathbb{R})\simeq\mathcal{M}(2^p,\mathbb{R}). \tag{3.40}$$

This result together with the relation **(iii)** in Eq. (3.38) and the Eq. (2.58), allow us to conclude that

$$\mathbb{H}\otimes\mathbb{H}\simeq\mathcal{C}\ell_{0,2}\otimes\mathcal{C}\ell_{0,2}\simeq\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,1}\simeq\mathcal{M}(2^2,\mathbb{R})\simeq\mathcal{M}(4,\mathbb{R}). \tag{3.41}$$

By Eq. (2.57) and the relations **(iii)** and **(iv)** in Eq. (3.38) we have that

$$\mathcal{C}\ell_{0,4}\simeq\mathcal{C}\ell_{0,2}\otimes\mathcal{C}\ell_{2,0}\simeq\mathbb{H}\otimes\mathcal{M}(2,\mathbb{R})\simeq\mathcal{M}(2,\mathbb{H})\simeq\mathcal{M}(2,\mathbb{R})\otimes\mathbb{H}\simeq\mathcal{C}\ell_{4,0}. \tag{3.42}$$

Together with those previous results, it follows from Eq. (2.60) that

$$\begin{aligned}\mathcal{C}\ell_{0,8}&\simeq\mathcal{C}\ell_{0,4}\otimes\mathcal{C}\ell_{0,4}\simeq\mathcal{M}(2,\mathbb{H})\otimes\mathcal{M}(2,\mathbb{H})\simeq\mathcal{M}(2,\mathbb{R})\otimes\mathbb{H}\otimes\mathbb{H}\otimes\mathcal{M}(2,\mathbb{R})\\&\simeq\mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(4,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{R})\simeq\mathcal{M}(16,\mathbb{R}).\end{aligned} \tag{3.43}$$

Now, using the above result (3.43) and Eq. (2.61) the following theorem has been proved.

Theorem 3.5. ***(Atiyah-Bott-Shapiro Periodicity Theorem)****. For every quadratic space* $\mathbb{R}^{p,q}$ *it follows that [5]*

$$\mathcal{C}\ell_{p,q+8}\simeq\mathcal{C}\ell_{p,q}\otimes\mathcal{M}(16,\mathbb{R}). \tag{3.44}$$

The Atiyah-Bott-Shapiro Periodicity Theorem 3.5 has a very important consequence, it tells us that we only need to explicitly obtain the classification of the Clifford algebras up to dim $V=p+q=8$, since for higher dimensions one can use the isomorphism $\mathcal{C}\ell_{p,q+8}\simeq\mathcal{C}\ell_{p,q}\otimes\mathcal{M}(16,\mathbb{R})$. By using all the previous results in this Chapter, we obtain

- $p+q=0$

  $\mathcal{C}\ell_{0,0}\simeq\mathbb{R}$

- $p+q=1$

  $\mathcal{C}\ell_{0,1}\simeq\mathbb{C}$

  $\mathcal{C}\ell_{1,0}\simeq\mathbb{R}\oplus\mathbb{R}$

- $p+q=2$

  $\mathcal{C}\ell_{0,2}\simeq\mathbb{H}$

$\mathcal{C}\ell_{2,0} \simeq \mathcal{M}(2,\mathbb{R})$

$\mathcal{C}\ell_{1,1} \simeq \mathcal{C}\ell_{2,0} \simeq \mathcal{M}(2,\mathbb{R})$

- ◇ $p+q=3$

$\mathcal{C}\ell_{0,3} \simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{1,0} \simeq \mathbb{H} \otimes (\mathbb{R} \oplus \mathbb{R}) \simeq \mathbb{H} \oplus \mathbb{H}$

$\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,1} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{C} \simeq \mathcal{M}(2,\mathbb{C})$

$\mathcal{C}\ell_{1,2} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{0,1} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{C} \simeq \mathcal{M}(2,\mathbb{C})$

$\mathcal{C}\ell_{2,1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{1,0} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{R} \oplus \mathbb{R} \simeq \mathcal{M}(2,\mathbb{R} \oplus \mathbb{R})$

- ◇ $p+q=4$

$\mathcal{C}\ell_{0,4} \simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{2,0} \simeq \mathbb{H} \otimes \mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(2,\mathbb{H})$

$\mathcal{C}\ell_{4,0} \simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{H} \simeq \mathcal{M}(2,\mathbb{H})$

$\mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{0,2} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{H} \simeq \mathcal{M}(2,\mathbb{H})$

$\mathcal{C}\ell_{3,1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{2,0} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(4,\mathbb{R})$

$\mathcal{C}\ell_{2,2} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{1,1} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(4,\mathbb{R})$

- ◇ $p+q=5$

$\mathcal{C}\ell_{0,5} \simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{3,0} \simeq \mathbb{H} \otimes \mathcal{M}(2,\mathbb{C}) \simeq \mathbb{H} \otimes \mathbb{C} \otimes \mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(2,\mathbb{C}) \otimes \mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(4,\mathbb{C})$

$\mathcal{C}\ell_{5,0} \simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,3} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{H} \oplus \mathbb{H} \simeq \mathcal{M}(2,\mathbb{H} \oplus \mathbb{H})$

$\mathcal{C}\ell_{1,4} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{0,3} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{H} \oplus \mathbb{H} \simeq \mathcal{M}(2,\mathbb{H} \oplus \mathbb{H})$

$\mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{3,0} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{C}) \simeq \mathcal{M}(4,\mathbb{C})$

$\mathcal{C}\ell_{2,3} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{1,2} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{0,1} \simeq \mathcal{M}(4,\mathbb{R}) \otimes \mathbb{C} \simeq \mathcal{M}(4,\mathbb{C})$

$\mathcal{C}\ell_{3,2} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{2,1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{1,0} \simeq \mathcal{M}(4,\mathbb{R}) \otimes \mathbb{R} \oplus \mathbb{R} \simeq \mathcal{M}(4,\mathbb{R} \oplus \mathbb{R})$

- ◇ $p+q=6$

$\mathcal{C}\ell_{0,6} \simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{4,0} \simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \simeq \mathbb{H} \otimes \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{H} \simeq \mathcal{M}(2,\mathbb{H}) \otimes \mathbb{H} \simeq \mathcal{M}(2,\mathbb{H} \otimes \mathbb{H})$

$$\simeq \mathcal{M}(2,\mathcal{C}\ell_{0,2}\otimes\mathcal{C}\ell_{0,2}) \simeq \mathcal{M}(2,\mathcal{C}\ell_{2,2}) \simeq \mathcal{M}(2,\mathcal{M}(4,\mathbb{R})) \simeq \mathcal{M}(8,\mathbb{R})$$

$$\mathcal{C}\ell_{6,0} \simeq \mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{0,4} \simeq \mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{0,2}\otimes\mathcal{C}\ell_{2,0} \simeq \mathcal{M}(2,\mathbb{R})\otimes\mathbb{H}\otimes\mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(4,\mathbb{H})$$

$$\mathcal{C}\ell_{1,5} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{0,4} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{0,2}\otimes\mathcal{C}\ell_{2,0} \simeq \mathcal{M}(2,\mathbb{R})\otimes\mathbb{H}\otimes\mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(4,\mathbb{H})$$

$$\mathcal{C}\ell_{5,1} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{4,0} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{0,2} \simeq \mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{R})\otimes\mathbb{H} \simeq \mathcal{M}(4,\mathbb{H})$$

$$\mathcal{C}\ell_{2,4} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{0,2} \simeq \mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{R})\otimes\mathbb{H} \simeq \mathcal{M}(4,\mathbb{H})$$

$$\mathcal{C}\ell_{4,2} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{3,1} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{2,0} \simeq \mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(8,\mathbb{R})$$

$$\mathcal{C}\ell_{3,3} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,1} \simeq \mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(8,\mathbb{R})$$

- $p+q=7$

$$\mathcal{C}\ell_{0,7} \simeq \mathcal{C}\ell_{0,2}\otimes\mathcal{C}\ell_{5,0} \simeq \mathbb{H}\otimes\mathcal{M}(2,\mathbb{H}\oplus\mathbb{H}) \simeq \mathcal{M}(2,(\mathbb{H}\otimes\mathbb{H})\oplus(\mathbb{H}\otimes\mathbb{H}))$$

$$\simeq \mathcal{M}(2,\mathbb{H}\otimes\mathbb{H})\oplus\mathcal{M}(2,\mathbb{H}\otimes\mathbb{H}) \simeq \mathcal{M}(8,\mathbb{R})\oplus\mathcal{M}(8,\mathbb{R}) \simeq \mathcal{M}(8,\mathbb{R}\oplus\mathbb{R})$$

$$\mathcal{C}\ell_{7,0} \simeq \mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{0,5} \simeq \mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(4,\mathbb{C}) \simeq \mathcal{M}(8,\mathbb{C})$$

$$\mathcal{C}\ell_{1,6} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{0,5} \simeq \mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(4,\mathbb{C}) \simeq \mathcal{M}(8,\mathbb{C})$$

$$\mathcal{C}\ell_{6,1} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{5,0} \simeq \mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{H}\oplus\mathbb{H}) \simeq \mathcal{M}(4,\mathbb{H}\oplus\mathbb{H})$$

$$\mathcal{C}\ell_{2,5} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,4} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{0,3} \simeq \mathcal{M}(4,\mathbb{R})\otimes\mathbb{H}\oplus\mathbb{H} \simeq \mathcal{M}(4,\mathbb{H}\oplus\mathbb{H})$$

$$\mathcal{C}\ell_{5,2} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{3,0} \simeq \mathcal{M}(4,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{C}) \simeq \mathcal{M}(8,\mathbb{C})$$

$$\mathcal{C}\ell_{3,4} \simeq \mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{3,0} \simeq \mathcal{M}(4,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{C}) \simeq \mathcal{M}(8,\mathbb{C})$$

$$\mathcal{C}\ell_{4,3} \simeq \mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{3,2} \simeq \mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{3,0} \simeq \mathcal{M}(8,\mathbb{R})\otimes\mathbb{R}\oplus\mathbb{R} \simeq \mathcal{M}(8,\mathbb{R}\oplus\mathbb{R})$$

Hence the following isomorphisms hold

$$\begin{aligned}
&\mathcal{C}\ell_{2,0} \simeq \mathcal{C}\ell_{1,1} \simeq \mathcal{M}(2,\mathbb{R})\\
&\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,2} \simeq \mathcal{M}(2,\mathbb{C})\\
&\mathcal{C}\ell_{0,4} \simeq \mathcal{C}\ell_{4,0} \simeq \mathcal{C}\ell_{1,3} \simeq \mathcal{M}(2,\mathbb{H})\\
&\mathcal{C}\ell_{3,1} \simeq \mathcal{C}\ell_{2,2} \simeq \mathcal{M}(4,\mathbb{R})\\
&\mathcal{C}\ell_{5,0} \simeq \mathcal{C}\ell_{1,4} \simeq \mathcal{M}(2,\mathbb{H}\oplus\mathbb{H})\\
&\mathcal{C}\ell_{0,5} \simeq \mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{2,3} \simeq \mathcal{M}(4,\mathbb{C})\\
&\mathcal{C}\ell_{6,0} \simeq \mathcal{C}\ell_{5,1} \simeq \mathcal{C}\ell_{1,5} \simeq \mathcal{C}\ell_{2,4} \simeq \mathcal{M}(4,\mathbb{H})\\
&\mathcal{C}\ell_{7,0} \simeq \mathcal{C}\ell_{1,6} \simeq \mathcal{C}\ell_{5,2} \simeq \mathcal{C}\ell_{3,4} \simeq \mathcal{M}(8,\mathbb{C})\\
&\mathcal{C}\ell_{0,7} \simeq \mathcal{C}\ell_{4,3} \simeq \mathcal{M}(8,\mathbb{R}\oplus\mathbb{R})\\
&\mathcal{C}\ell_{6,1} \simeq \mathcal{C}\ell_{2,5} \simeq \mathcal{M}(4,\mathbb{H}\oplus\mathbb{H})
\end{aligned}$$

Suppose that $p > q$ and take $p - q = 8k + r$ with $r < 8$. One can use the relations shown in the Theorem 2.14 and Eq. (2.56) to obtain:

$$\begin{aligned}
\mathcal{C}\ell_{p,q} &\simeq \mathcal{C}\ell_{q,q} \otimes \mathcal{C}\ell_{p-q,0}\\
&\simeq \mathcal{C}\ell_{q,q} \otimes \mathcal{C}\ell_{8k+r,0}\\
&\simeq \mathcal{C}\ell_{q,q} \otimes \mathcal{C}\ell_{(8(k-1)+r+6)+2,0}\\
&\simeq \mathcal{C}\ell_{q,q} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,8(k-1)+r+6}.
\end{aligned} \tag{3.45}$$

Recalling the Eq. (2.61), a similar relation can be obtained as consequence of Corollary 2.16, that is

$$\mathcal{C}\ell_{p,q+8k'} \simeq \mathcal{C}\ell_{0,8k'} \otimes \mathcal{C}\ell_{p,q} \tag{3.46}$$

for any $k' > 0$, therefore, we have

$$\mathcal{C}\ell_{0,8(k-1)+r+6} \simeq \mathcal{C}\ell_{0,8(k-1)} \otimes \mathcal{C}\ell_{0,r+6}. \tag{3.47}$$

We also have that

$$\begin{aligned}
\mathcal{C}\ell_{0,r+6} &\simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{r,0}\\
&\simeq \mathcal{C}\ell_{2,2} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{r,0}
\end{aligned} \tag{3.48}$$

Hence,

$$\begin{aligned}\mathcal{C}\ell_{p,q} &\simeq \mathcal{C}\ell_{q,q}\otimes\mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{0,8(k-1)+r+6}\\ &\simeq \mathcal{C}\ell_{q,q}\otimes\mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{0,8(k-1)}\otimes\mathcal{C}\ell_{2,2}\otimes\mathcal{C}\ell_{2,0}\otimes\mathcal{C}\ell_{r,0}\\ &\simeq \mathcal{M}(2^q,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(16^{k-1},\mathbb{R})\otimes\mathcal{M}(4,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{R})\otimes\mathcal{C}\ell_{r,0}\\ &\simeq \mathcal{M}(2^q,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{R})\otimes\mathcal{M}(2^{4(k-1)},\mathbb{R})\otimes\mathcal{M}(2^2,\mathbb{R})\otimes\mathcal{M}(2,\mathbb{R})\otimes\mathcal{C}\ell_{r,0}\\ &\simeq \mathcal{M}(2^{q+4k},\mathbb{R})\otimes\mathcal{C}\ell_{r,0}\end{aligned}\tag{3.49}$$

Now, if $q > p$ and by considering $q-p = 8k+r$, it yields [7]

$$\begin{aligned}\mathcal{C}\ell_{p,q} &\simeq \mathcal{C}\ell_{p,p}\otimes\mathcal{C}\ell_{0,q-p}\simeq\mathcal{C}\ell_{p,p}\otimes\mathcal{C}\ell_{0,8k+r}\simeq\mathcal{C}\ell_{p,p}\otimes\mathcal{C}\ell_{0,8k}\mathcal{C}\ell_{0,r}\\ &\simeq \mathcal{M}(2^p,\mathbb{R})\otimes\mathcal{M}(2^{4k},\mathbb{R})\otimes\mathcal{C}\ell_{0,r}\simeq\mathcal{M}(2^{p+4k},\mathbb{R})\otimes\mathcal{C}\ell_{0,r}\end{aligned}\tag{3.50}$$

Therefore, the Clifford algebra is determined by $r = p-q \mod 8$. One may analyse the possibilities as follows.

- $r=0$
    - ◇ $p \geq q\ (p-q \mod 8 = 0):$

$$\mathcal{C}\ell_{p,q}\simeq\mathcal{M}(2^{q+4k},\mathbb{R})\otimes\mathcal{C}\ell_{0,0}\simeq\mathcal{M}(2^{q+4k},\mathbb{R})$$

    - ◇ $p < q\ (p-q \mod 8 = 0):$

$$\mathcal{C}\ell_{p,q}\simeq\mathcal{M}(2^{p+4k},\mathbb{R})\otimes\mathcal{C}\ell_{0,0}\simeq\mathcal{M}(2^{p+4k},\mathbb{R})$$

- $r=1$
    - ◇ $p > q\ (p-q \mod 8 = 1):$

$$\begin{aligned}\mathcal{C}\ell_{p,q} &\simeq \mathcal{M}(2^{q+4k},\mathbb{R})\otimes\mathcal{C}\ell_{1,0}\\ &\simeq \mathcal{M}(2^{q+4k},\mathbb{R})\otimes(\mathbb{R}\oplus\mathbb{R})\simeq\mathcal{M}(2^{q+4k},\mathbb{R})\oplus\mathcal{M}(2^{q+4k},\mathbb{R})\end{aligned}$$

    - ◇ $p < q\ (q-p \mod 8 = 1 \Rightarrow p-q \mod 8 = 7):$

$$\mathcal{C}\ell_{p,q}\simeq\mathcal{M}(2^{p+4k},\mathbb{R})\otimes\mathcal{C}\ell_{0,1}\simeq\mathcal{M}(2^{p+4k},\mathbb{R})\otimes\mathbb{C}\simeq\mathcal{M}(2^{p+4k},\mathbb{C})$$

  In the above cases, $q+4k = \left[\frac{2q+8k+r}{2}\right] = \left[\frac{p+q}{2}\right] = \left[\frac{n}{2}\right]$, where $[s]$ denotes the integer part of s.

- $r=2$

- ◇ $p > q \ (p - q \mod 8 = 2)$:

$$\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{q+4k}, \mathbb{R}) \otimes \mathcal{C}\ell_{2,0} \simeq \mathcal{M}(2^{q+4k}, \mathbb{R}) \otimes \mathcal{M}(2, \mathbb{R}) \simeq \mathcal{M}(2^{q+4k+1}, \mathbb{R})$$

- ◇ $p < q \ (p - q \mod 8 = 6)$:

$$\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{p+4k}, \mathbb{R}) \otimes \mathcal{C}\ell_{0,2} \simeq \mathcal{M}(2^{p+4k}, \mathbb{R}) \otimes \mathbb{H} \simeq \mathcal{M}(2^{p+4k}, \mathbb{H})$$

- $r = 3$
  - ◇ $p > q \ (p - q \mod 8 = 3)$:

$$\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{q+4k}, \mathbb{R}) \otimes \mathcal{C}\ell_{3,0} \simeq \mathcal{M}(2^{q+4k}, \mathbb{R}) \otimes \mathcal{M}(2, \mathbb{C}) \simeq \mathcal{M}(2^{q+4k+1}, \mathbb{C})$$

  - ◇ $p < q \ (p - q \mod 8 = 5)$:

$$\begin{aligned} \mathcal{C}\ell_{p,q} &\simeq \mathcal{M}(2^{p+4k}, \mathbb{R}) \otimes \mathcal{C}\ell_{0,3} \\ &\simeq \mathcal{M}(2^{p+4k}, \mathbb{R}) \otimes (\mathbb{H} \oplus \mathbb{H}) \simeq \mathcal{M}(2^{p+4k}, \mathbb{H}) \oplus \mathcal{M}(2^{p+4k}, \mathbb{H}) \end{aligned}$$

In the cases where $r = 2, 3$, if $p > q$ then $q + 4k + 1 = \left[\frac{2q+8k+r}{2}\right] = \left[\frac{n}{2}\right]$ and if $p < q$, $p + 4k = \left[\frac{2p+8k+r}{2}\right] - 1 = \left[\frac{n}{2}\right] - 1$.

- $r = 4$
  - ◇ $p > q \ (p - q \mod 8 = 4)$:

$$\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{q+4k}, \mathbb{R}) \otimes \mathcal{C}\ell_{4,0} \simeq \mathcal{M}(2^{q+4k}, \mathbb{R}) \otimes \mathcal{M}(2, \mathbb{H}) \simeq \mathcal{M}(2^{q+4k+1}, \mathbb{H})$$

  - ◇ $p < q \ (p - q \mod 8 = 4)$:

$$\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{p+4k}, \mathbb{R}) \otimes \mathcal{C}\ell_{0,4} \simeq \mathcal{M}(2^{p+4k}, \mathbb{R}) \otimes \mathcal{M}(2, \mathbb{H}) \simeq \mathcal{M}(2^{p+4k+1}, \mathbb{H})$$

- $r = 5$
  - ◇ $p > q \ (p - q \mod 8 = 5)$:

$$\begin{aligned} \mathcal{C}\ell_{p,q} &\simeq \mathcal{M}(2^{q+4k}, \mathbb{R}) \otimes \mathcal{C}\ell_{5,0} \simeq \mathcal{M}(2^{q+4k}, \mathbb{R}) \otimes \mathcal{M}(2, \mathbb{R}) \otimes (\mathbb{H} \oplus \mathbb{H}) \\ &\simeq \mathcal{M}(2^{q+4k}, \mathbb{R}) \otimes (\mathcal{M}(2, \mathbb{H}) \oplus \mathcal{M}(2, \mathbb{H})) \\ &\simeq \mathcal{M}(2^{q+4k+1}, \mathbb{H}) \oplus \mathcal{M}(2^{q+4k+1}, \mathbb{H}). \end{aligned}$$

- ◇ $p<q\ (p-q \mod 8=3):$

$$\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{p+4k},\mathbb{R})\otimes\mathcal{C}\ell_{0,5} \simeq \mathcal{M}(2^{p+4k},\mathbb{R})\otimes\mathcal{M}(4,\mathbb{C}) \simeq \mathcal{M}(2^{p+4k+2},\mathbb{C})$$

In the cases where $r=4,5$, if $p>q$ or $r=4$ then $q+4k+1=\left[\frac{n}{2}\right]-1$. If $p<q$ and $r=5$, then $p+4k+2=\left[\frac{n}{2}\right]$.

- $r=6$
  - ◇ $p>q\ (p-q \mod 8=6):$

$$\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{q+4k},\mathbb{R})\otimes\mathcal{C}\ell_{6,0} \simeq \mathcal{M}(2^{q+4k},\mathbb{R})\otimes\mathcal{M}(4,\mathbb{H}) \simeq \mathcal{M}(2^{q+4k+2},\mathbb{H})$$

  - ◇ $p<q\ (p-q \mod 8=2):$

$$\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{p+4k},\mathbb{R})\otimes\mathcal{C}\ell_{0,6} \simeq \mathcal{M}(2^{p+4k},\mathbb{R})\otimes\mathcal{M}(8,\mathbb{R}) \simeq \mathcal{M}(2^{p+4k+3},\mathbb{C})$$

- $r=7$
  - ◇ $p>q\ (p-q \mod 8=7):$

$$\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{q+4k},\mathbb{R})\otimes\mathcal{C}\ell_{7,0} \simeq \mathcal{M}(2^{q+4k},\mathbb{R})\otimes\mathcal{M}(8,\mathbb{C}) \simeq \mathcal{M}(2^{q+4k+3},\mathbb{C})$$

  - ◇ $p<q\ (p-q \mod 8=1):$

$$\begin{aligned}\mathcal{C}\ell_{p,q} &\simeq \mathcal{M}(2^{p+4k},\mathbb{R})\otimes\mathcal{C}\ell_{0,7} \simeq \mathcal{M}(2^{p+4k},\mathbb{R})\otimes\mathcal{M}(8,\mathbb{R}\oplus\mathbb{R})\\ &\simeq \mathcal{M}(2^{p+4k},\mathbb{R})\otimes\mathcal{M}(8,\mathbb{R})\otimes(\mathbb{R}\oplus\mathbb{R})\\ &\simeq \mathcal{M}(2^{p+4k+3},\mathbb{R})\oplus\mathcal{M}(2^{p+4k+3},\mathbb{R})\end{aligned}$$

In the cases where $r=6,7$, if $p<q$, therefore $p+4k+3=\left[\frac{n}{2}\right]$; and if $p>q$ and $r=6$, hence $q+4k+2=\left[\frac{n}{2}\right]-1$.

Right, one can organise the algebras of dimension $n<8$ according to $p-q$, obtaining

$$\begin{array}{lll}
p-q=0: & \mathcal{C}\ell_{0,0},\ \mathcal{C}\ell_{1,1},\ \mathcal{C}\ell_{2,2},\ \mathcal{C}\ell_{3,3} & \mathcal{M}(2^{[\frac{n}{2}]},\mathbb{R}) \\
p-q=1: & \mathcal{C}\ell_{1,0},\ \mathcal{C}\ell_{2,1},\ \mathcal{C}\ell_{3,2},\ \mathcal{C}\ell_{4,3},\ \mathcal{C}\ell_{0,7} & \mathcal{M}(2^{[\frac{n}{2}]},\mathbb{R})\oplus\mathcal{M}(2^{[\frac{n}{2}]},\mathbb{R}) \\
p-q=2: & \mathcal{C}\ell_{2,0},\ \mathcal{C}\ell_{3,1},\ \mathcal{C}\ell_{4,2},\ \mathcal{C}\ell_{0,6} & \mathcal{M}(2^{[\frac{n}{2}]},\mathbb{R}) \\
p-q=3: & \mathcal{C}\ell_{3,0},\ \mathcal{C}\ell_{4,1},\ \mathcal{C}\ell_{5,2},\ \mathcal{C}\ell_{0,5},\ \mathcal{C}\ell_{1,6} & \mathcal{M}(2^{[\frac{n}{2}]},\mathbb{C}) \\
p-q=4: & \mathcal{C}\ell_{4,0},\ \mathcal{C}\ell_{5,1},\ \mathcal{C}\ell_{0,4},\ \mathcal{C}\ell_{1,5} & \mathcal{M}(2^{[\frac{n}{2}]-1},\mathbb{H}) \\
p-q=5: & \mathcal{C}\ell_{5,0},\ \mathcal{C}\ell_{6,1},\ \mathcal{C}\ell_{0,3},\ \mathcal{C}\ell_{1,4},\ \mathcal{C}\ell_{2,5} & \mathcal{M}(2^{[\frac{n}{2}]-1},\mathbb{H})\oplus\mathcal{M}(2^{[\frac{n}{2}]-1},\mathbb{H}) \\
p-q=6: & \mathcal{C}\ell_{6,0},\ \mathcal{C}\ell_{0,2},\ \mathcal{C}\ell_{1,3},\ \mathcal{C}\ell_{2,4} & \mathcal{M}(2^{[\frac{n}{2}]-1},\mathbb{H}) \\
p-q=7: & \mathcal{C}\ell_{7,0},\ \mathcal{C}\ell_{0,1},\ \mathcal{C}\ell_{1,2},\ \mathcal{C}\ell_{2,3},\ \mathcal{C}\ell_{3,4} & \mathcal{M}(2^{[\frac{n}{2}]},\mathbb{C})
\end{array}$$

Therefore, we classify the real Clifford algebras over $\mathbb{R}^{p,q}$ as

$p-q \mod 8$	$\mathcal{C}\ell_{p,q}$
0	$\mathcal{M}(2^{[\frac{n}{2}]},\mathbb{R})$
1	$\mathcal{M}(2^{[\frac{n}{2}]},\mathbb{R})\oplus\mathcal{M}(2^{[\frac{n}{2}]},\mathbb{R})$
2	$\mathcal{M}(2^{[\frac{n}{2}]},\mathbb{R})$
3	$\mathcal{M}(2^{[\frac{n}{2}]},\mathbb{C})$
4	$\mathcal{M}(2^{[\frac{n}{2}]-1},\mathbb{H})$
5	$\mathcal{M}(2^{[\frac{n}{2}]-1},\mathbb{H})\oplus\mathcal{M}(2^{[\frac{n}{2}]-1},\mathbb{H})$
6	$\mathcal{M}(2^{[\frac{n}{2}]-1},\mathbb{H})$
7	$\mathcal{M}(2^{[\frac{n}{2}]},\mathbb{C})$

Table 3.1.: Real Clifford Algebras Classification

Example 3.6 ▹ Real Clifford algebras classification table usage.

Let us illustrate the usage of the classification table. Consider the Clifford algebra $\mathcal{C}\ell_{3,0}$, where $p-q=3$, by looking at the above table it can be seen that it is isomorphic to $\mathcal{M}(2^{[\frac{n}{2}]},\mathbb{C})$. Since $n=p+q=3$ and $\left[\frac{n}{2}\right]=\left[\frac{3}{2}\right]=1$, we can conclude that $\mathcal{C}\ell_{3,0}\simeq\mathcal{M}(2^1,\mathbb{C})=\mathcal{M}(2,\mathbb{C})$. Consider now the algebra $\mathcal{C}\ell_{0,2}$. In this case $p-q=-2=6$ mod 8 and the corresponding algebra is $\mathcal{M}(2^{[\frac{n}{2}]-1},\mathbb{H})$. Since $n=p+q=2$ and $\left[\frac{n}{2}\right]=\left[\frac{2}{2}\right]=1$ it follows that $\mathcal{C}\ell_{0,2}\simeq\mathcal{M}(2^{1-1},\mathbb{H})=\mathcal{M}(1,\mathbb{H})=\mathbb{H}$ as already seen previously. ◃

With respect to the complex case the classification can be obtained by the relation $\mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) \simeq \mathbb{C} \otimes \mathcal{C}\ell(V, g)$ shown in Theorem 2.13. We denote $\mathbb{C} \otimes \mathcal{C}\ell(V, g)$ as $\mathcal{C}\ell_{\mathbb{C}}(n)$. It follows that [7]

$$\begin{aligned}
p-q=0: &\quad \mathbb{C}\otimes\mathcal{M}(2^n,\mathbb{R}) \simeq \mathcal{M}(2^n,\mathbb{C})\\
p-q=1: &\quad \mathbb{C}\otimes\mathcal{M}(2^n,\mathbb{R})\oplus\mathcal{M}(2^n,\mathbb{R}) \simeq \mathcal{M}(2^n,\mathbb{C})\oplus\mathcal{M}(2^n,\mathbb{C})\\
p-q=2: &\quad \mathbb{C}\otimes\mathcal{M}(2^n,\mathbb{R}) \simeq \mathcal{M}(2^n,\mathbb{C})\\
p-q=3: &\quad \mathbb{C}\otimes\mathcal{M}(2^n,\mathbb{C}) \simeq \mathcal{M}(2^n,\mathbb{C})\oplus\mathcal{M}(2^n,\mathbb{C})\\
p-q=4: &\quad \mathbb{C}\otimes\mathcal{M}(2^{n-1},\mathbb{H}) \simeq \mathcal{M}(2^n,\mathbb{C})\\
p-q=5: &\quad \mathbb{C}\otimes\mathcal{M}(2^{n-1},\mathbb{H})\oplus\mathcal{M}(2^{n-1},\mathbb{H}) \simeq \mathcal{M}(2^n,\mathbb{C})\oplus\mathcal{M}(2^n,\mathbb{C})\\
p-q=6: &\quad \mathbb{C}\otimes\mathcal{M}(2^{n-1},\mathbb{H}) \simeq \mathcal{M}(2^n,\mathbb{C})\\
p-q=7: &\quad \mathbb{C}\otimes\mathcal{M}(2^n,\mathbb{C}) \simeq \mathcal{M}(2^n,\mathbb{C})\oplus\mathcal{M}(2^n,\mathbb{C})
\end{aligned}$$

As we have seen, the complex Clifford algebra $\mathcal{C}\ell_{\mathbb{C}}(n)$ depends only on the parity of $n = p+q$. Hence, the complex Clifford algebra classification is given by the following table

n even	$\mathcal{C}\ell_{\mathbb{C}}(2k) \simeq \mathcal{M}(2^k,\mathbb{C})$
n odd	$\mathcal{C}\ell_{\mathbb{C}}(2k+1) \simeq \mathcal{M}(2^k,\mathbb{C})\oplus\mathcal{M}(2^k,\mathbb{C})$

Table 3.2.: Complex Clifford Algebras Classification

CONCLUSION

The definition of the Clifford algebras was introduced together with their construction as a quotient of the tensor algebra by a two-sided ideal. The main theorems of its structure have been analysed and the implementation of the Clifford algebra classification based on representation theory was done successfully achieving the main objective of this work.

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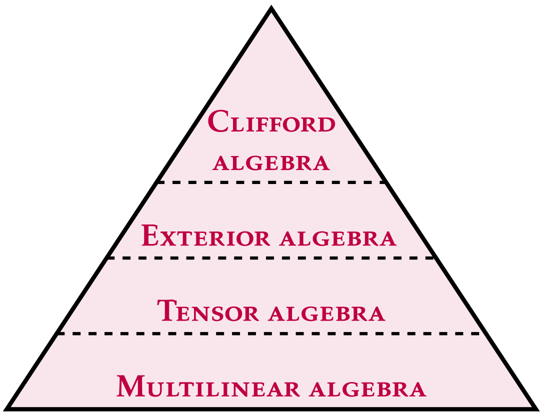
A Multilinear Algebra

The basis of Clifford algebra is the multilinear algebra. This Appendix A contains a summary of certain topics in multilinear algebra that are required for the sequel. We briefly introduce some concepts on multilinear algebra, set up notation and terminology about multilinear mappings and quadratics spaces. Our aim here is to establish the fundamental structures for a Clifford algebra to be defined, namely: quadratic spaces, inner product, musical isomorphism. In the next Appendix B, we proceed with the study of the tensor algebra, we review some of the standard definitions of the tensor product, give a formal construction of the tensor product spaces and then we finish presenting the tensor algebra that is important to construc a Clifford algebra. Finally, in the Appendix C, taking into account the concepts developed previously, we will be concerned with the exterior algebra and its properties which is very important to the Clifford algebra , since it carries the multivector structure of the exterior algebra.



Definition A.1. *Let $V_1, V_2, \dots, V_p$ and W be a finite family of vector spaces over the same field $\mathbb{K}$. The mapping*

$$\varphi : V_1 \times V_2 \times \cdots \times V_p \to W \tag{A.1}$$

*is called a **multilinear mapping** (in this case p-linear) if it is linear in each argument, when the others are fixed, which means that given arbitrary $\lambda \in \mathbb{K}$, $\mathbf{v}_1, \mathbf{v}'_1 \in V_1$, $\mathbf{v}_2, \mathbf{v}'_2 \in$*

$V_2, \dots, \mathbf{v}_n, \mathbf{v}'_n \in V_n$, then

$$\begin{aligned} &\textbf{(i)}\ \varphi(\mathbf{v}_1,\dots,\mathbf{v}_i,\dots,\mathbf{v}_p)+\varphi(\mathbf{v}_1,\dots,\mathbf{v}'_i,\dots,\mathbf{v}_p)=\varphi(\mathbf{v}_1,\dots,\mathbf{v}_i+\mathbf{v}'_i,\dots,\mathbf{v}_p),\\ &\textbf{(ii)}\ \lambda\varphi(\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_i,\dots,\mathbf{v}_p)=\varphi(\mathbf{v}_1,\mathbf{v}_2,\dots,\lambda\mathbf{v}_i,\dots,\mathbf{v}_p) \end{aligned} \tag{A.2}$$

for $i = 1, .., p$. When $p = 1$ the mapping is said to be linear and when $p = 2$, bilinear.

We denote the vector space of such multilinear mappings by $\mathcal{L}(V_1, V_2, \dots, V_p; W)$. In our work, it is necessary to consider a symmetric bilinear mapping and its quadratic form. A bilinear mapping $g : V \times V \to \mathbb{K}$ is said to be *symmetric* if for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ the property $g(\mathbf{v}_1, \mathbf{v}_2) = g(\mathbf{v}_2, \mathbf{v}_1)$ holds. A bilinear form is said to be *non-degenerate* if

$$\begin{aligned} &g(\mathbf{v}_1,\mathbf{v}_2)=0 \text{ for all } \mathbf{v}_2\in V \text{ implies } \mathbf{v}_1=0 \text{ and}\\ &g(\mathbf{v}_1,\mathbf{v}_2)=0 \text{ for all } \mathbf{v}_1\in V \text{ implies } \mathbf{v}_2=0. \end{aligned} \tag{A.3}$$

Definition A.2. *Let V a vector space over the field $\mathbb{K}$. A* ***quadratic form*** *Q on the space V is a mapping $Q : V \to \mathbb{K}$ for which there exists a bilinear form $g : V \times V \to \mathbb{K}$ such that for all $\mathbf{v} \in V$*

$$Q(\mathbf{v}) = g(\mathbf{v}, \mathbf{v}). \tag{A.4}$$

If the characteristic of the field $\mathbb{K}$ is different of 2, then for any quadratic form Q there exists a unique symmetric bilinear form g with the property $Q(\mathbf{v}) = g(\mathbf{v}, \mathbf{v})$, called the *polarisation* of Q [2]. In terms of its quadratic form $Q : V \to \mathbb{K}$ the bilinear form g can be expressed as

$$g(\mathbf{v}, \mathbf{w}) = \frac{1}{2}(Q(\mathbf{v}+\mathbf{w}) - Q(\mathbf{v}) - Q(\mathbf{w})). \tag{A.5}$$

for every $\mathbf{v}, \mathbf{w} \in V$. A vector space V equipped with a symmetric bilinear mapping $g : V \times V \to \mathbb{K}$ is said to be a **quadratic space** which is the main structure on which we define a Clifford algebra.

Example 1.3 ▹ Inner product.

In a vector space V an inner product $\langle\ ,\ \rangle : V \times V \to \mathbb{K}$ is a positive definite symmetric bilinear form, which means that, for every $\mathbf{v} \in V$, $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, whereas the equality holds if and only if $\mathbf{v} = 0$. The standard inner product in $\mathbb{R}^n$ for $\mathbf{v} = (v_1, \dots, v_n)$ and

$\mathbf{w} = (w_1,\dots,w_n) \in \mathbb{R}^n$ is given by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v_i w_i. \quad \triangleleft \tag{A.6}$$

Example 1.4 ▹ Signature of a general bilinear form

Let V a vector space over $\mathbb{R}$ with basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$. For $0 \leq p \leq 0$ and $0 \leq r \leq 0$ and $\alpha_1,\dots,\alpha_n, \alpha_{p+r+1},\dots,\alpha_n \in \mathbb{R}_+^*$ some bilinear form f can be defined more generally as

$$f(\mathbf{v},\mathbf{w}) = -\sum_{i=1}^{p} \alpha_i v_i w_i + \sum_{i=p+1}^{p+q} \alpha_i v_i w_i + \sum_{i=p+q+1}^{n} 0 v_i w_i. \tag{A.7}$$

For $\mathbf{v} = (v_1,\dots,v_n), \mathbf{w} = (w_1,\dots,w_n) \in V$. If $r = n-(p+q) \neq 0$ the bilinear form is degenerate, the **signature** of the bilinear form is the numbers (p,q,r). In the non-degenerate case $(r = 0)$ the signature of the bilinear form is (p,q). ◃

A.1. Musical Isomorphisms

Let V be a vector space and let us consider its dual space denoted by V^*. There does not exist a natural isomorphism between those spaces like there exists for V and its bidual space $(V^*)^*$ [7]. An additional structure is required to define the isomorphism $V \simeq V^*$ called *correlation*. In other words, an isomorphism between V and V^* must to be chosen. One can introduce the correlation as the linear mapping $\tau : V \to V^*$ that naturally defines a bilinear functional $B : V \times V \to \mathbb{R}$ by the equation

$$B(\mathbf{v},\mathbf{u}) = \tau(\mathbf{v})(\mathbf{u}), \quad \text{for all } \mathbf{u},\mathbf{v} \in V. \tag{A.8}$$

Since $\dim V = \dim V^*$, if τ is non-degenerated, i.e., $\ker \tau = \{0\}$, it follows that τ is an *isomorphism*. In the case of quadratic spaces, we denote the correlations $\tau : V \to V^*$ and $\tau^{-1} : V^* \to V$ by

$$\flat : V \to V^*, \quad \sharp : V^* \to V \tag{A.9}$$

in such way $\flat = \sharp^{-1}$ and $\sharp = \flat^{-1}$. Such isomorphisms are called **musical isomorphisms** with respect to g. Alternatively, it is also used the expression

$$\mathbf{v}_\flat = \flat(\mathbf{v}), \quad \alpha^\sharp = \sharp(\alpha). \tag{A.10}$$

Hence by definition it follows that

$$\mathbf{v}_\flat(\mathbf{u}) = g(\mathbf{v}, \mathbf{u}), \quad g(\alpha^\sharp, \mathbf{v}) = \alpha(\mathbf{v}). \tag{A.11}$$

The musical isomorphism is important to characterise the Clifford product on Clifford algebras.

B TENSOR ALGEBRA

The theory of tensor algebra is fundamental to our work, through tensor algebra we are able to construct the exterior algebra and the Clifford algebra. In addition, the tensor product of spaces is considered oftentimes in theorems concerning the Clifford algebra structure. Hence, this appendix is devoted to establishing some notations, concepts and results about the tensor product together with some motivations and a discussion about it.

B.1. Tensor Product

The tensor product is defined in the reference [8] as follows

Definition B.1. *Let U and V vector spaces over the same field $\mathbb{K}$. The* ***tensor product*** *between U and V is a vector space T with a bilinear mapping*

$$\begin{aligned} \otimes : U \times V &\to T \\ (\mathbf{u},\mathbf{v}) &\mapsto (\mathbf{u}\otimes\mathbf{v}) \end{aligned} \tag{B.1}$$

satisfying the following condition: if $\{\mathbf{e}_1,\ldots,\mathbf{e}_m\}$ and $\{\mathbf{f}_1,\ldots,\mathbf{f}_n\}$ are basis for U and V respectively, then $\mathbf{e}_i\otimes\mathbf{f}_j$, $1\le i\le m$, $1\le j\le n$ is a basis for T.

Such condition in Definition B.1 does not depend on the choice of basis for U and V and can be expressed in the form of table as

$\otimes$	$\mathbf{f}_1$	$\mathbf{f}_2$	$\cdots$	$\mathbf{f}_n$
$\mathbf{e}_1$	$\mathbf{e}_1\otimes\mathbf{f}_1$	$\mathbf{e}_1\otimes\mathbf{f}_2$	$\cdots$	$\mathbf{e}_1\otimes\mathbf{f}_n$
$\mathbf{e}_2$	$\mathbf{e}_2\otimes\mathbf{f}_1$	$\mathbf{e}_2\otimes\mathbf{f}_2$	$\cdots$	$\mathbf{e}_2\otimes\mathbf{f}_n$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\mathbf{e}_m$	$\mathbf{e}_m\otimes\mathbf{f}_1$	$\mathbf{e}_m\otimes\mathbf{f}_2$	$\cdots$	$\mathbf{e}_m\otimes\mathbf{f}_n$

Therefore, $\dim U\otimes V = \dim U \dim V$. The consequence about this fact is that not every tensor $x\in U\otimes V$ has the form of a pure tensor $x=\mathbf{u}\otimes\mathbf{v}$ for some $\mathbf{u}\in U$ and $\mathbf{v}\in V$, nevertheless, since the dimension of $U\otimes V$ is the product of the dimensions of U and V not the sum of them, it follows that one general element from $U \otimes V$ is not a pure tensor, but rather a finite linear combination of pure tensors. In this way, the vector space $U \otimes V$ is the vector space of linear combinations of elements $\mathbf{u} \otimes \mathbf{v}$, $\mathbf{u} \in U$ and $\mathbf{v} \in V$ satisfying

$$\begin{aligned} (\mathbf{u}_1 + \mathbf{u}_2) \otimes \mathbf{v} &= \mathbf{u}_1 \otimes \mathbf{v} + \mathbf{u}_2 \otimes \mathbf{v}, \\ \mathbf{u} \otimes (\mathbf{v}_1 + \mathbf{v}_2) &= \mathbf{u} \otimes \mathbf{v}_1 + \mathbf{u} \otimes \mathbf{v}_2, \\ (\lambda \mathbf{u}) \otimes \mathbf{v} &= \mathbf{u} \otimes (\lambda \mathbf{v}) = \lambda(\mathbf{u} \otimes \mathbf{v}), \end{aligned} \tag{B.2}$$

for $\lambda \in \mathbb{K}$, $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2 \in U$, $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V$, since $\otimes$ is a bilinear mapping. Right, it is easier to understand the motivation about the tensor product once it has been defined, so now we are in a position to present a motivation behind the tensor product through multilinear mappings and another definition that follows from it.

With respect to linear mappings we know that the composition of two linear mappings is a linear mapping, however, it is not true that the composition of two multilinear mappings is still a multilinear mapping. For instance, let us consider the following two multilinear mappings

$$\begin{aligned} f &: V_1 \times V_2 \to W_1 \times W_2 \times W_3, \\ g &: W_1 \times W_2 \times W_3 \to U. \end{aligned} \tag{B.3}$$

We have that f is bilinear and g is trilinear, but what about the composition $g \circ f$?

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & W_1 \times W_2 \times W_3 \arrow[r, "g"] & U \end{tikzcd}$$

Let us investigate what happens with respect to that composition, writing f as

$$f(\mathbf{v}_1, \mathbf{v}_2) = (f_1(\mathbf{v}_1, \mathbf{v}_2), f_2(\mathbf{v}_1, \mathbf{v}_2), f_3(\mathbf{v}_1, \mathbf{v}_2)) \tag{B.4}$$

for $\mathbf{v}_1 \in V_1, \mathbf{v}_2 \in V_2$ we have that since f is bilinear, so is its components f_1, f_2, f_2. Now, inspecting $g \circ f(\lambda \mathbf{v}_1, \mathbf{v}_2)$ for $\lambda \in \mathbb{K}$ we have that

$$\begin{aligned} g \circ f(\lambda \mathbf{v}_1, \mathbf{v}_2) &= g(f(\lambda \mathbf{v}_1, \mathbf{v}_2)) \\ &= g(f_1(\lambda \mathbf{v}_1, \mathbf{v}_2), f_2(\lambda \mathbf{v}_1, \mathbf{v}_2), f_3(\lambda \mathbf{v}_1, \mathbf{v}_2)) \\ &= g((\lambda f_1(\mathbf{v}_1, \mathbf{v}_2), \lambda f_2(\mathbf{v}_1, \mathbf{v}_2), \lambda f_3(\mathbf{v}_1, \mathbf{v}_2)) \\ &= \lambda^3 g(f_1(\mathbf{v}_1, \mathbf{v}_2), f_2(\mathbf{v}_1, \mathbf{v}_2), f_3(\mathbf{v}_1, \mathbf{v}_2)) \\ &= \lambda^3 g \circ f(\mathbf{v}_1, \mathbf{v}_2). \end{aligned} \tag{B.5}$$

Therefore, linearity or multilinearity does not hold for $g \circ f$. For that reason, we conclude that the multilinearity is not a well-behaved condition, which motivates us to construct a new vector space, starting from the spaces that we are already considering, with the property that for every multilinear mapping on these product spaces there exists a unique linear mapping on this new vector space.

Definition B.2. *Given two vector spaces U and V, the **tensor product** between these spaces is a new vector space $U \otimes V$ and a bilinear mapping $\otimes : U \times V \to U \otimes V$ such that for any other bilinear mapping $f : U \times V \to W$, where W is a vector space, there exists a unique linear mapping $\overline{f} : U \otimes V \to W$ such that $f = \overline{f} \circ \otimes$.*

$$\begin{tikzcd}
U \times V \arrow[r, "\otimes"] \arrow[dr, "f"'] & U \otimes V \arrow[d, dashed, "\overline{f}"] \\
 & W
\end{tikzcd}$$

The aim of the Definition B.2 is that the new vector space $U \otimes V$ linearises the bilinear mappings out of $U \times V$ and owns the information of the original spaces as well. In addition, the Definition B.2 establish $\mathcal{L}(U \otimes V, W) \simeq \mathcal{L}(U, V; W)$ such that $(\overline{f} : U \otimes V \to W) \mapsto (f : U \times V \to W)$. If $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ and $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ are basis for U and V respectively, then the linear mapping $\overline{f}$ is given by [8]

$$\overline{f}(\mathbf{e}_i \otimes \mathbf{f}_j) = f(\mathbf{e}_i, \mathbf{f}_j). \tag{B.6}$$

Furthermore, the condition shown in the Definition B.2 is called universality, such property is quite common in algebra

Example 2.3 ▹ Homomorphism theorem

Let V a vector space, $L \subset V$ vector subspace over $\mathbb{K}$ and $\pi : V \to V/L$ the quotient mapping then for any linear mapping $\varphi : V \to W$ satisfying $L \subset \ker\varphi$ there exists a unique linear mapping $\overline{\varphi} : V/L \to W$ such that $\overline{\varphi} \circ \pi = \varphi$ [9]

$$\begin{tikzcd}
V \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & V/L \arrow[d, dashed, "\overline{\varphi}"] \\
 & W
\end{tikzcd}$$

i.e., the above diagram commutes. ◃

The universality of the tensor product has several benefits, the universality guarantees the uniqueness of the tensor product, in situations when we desire to prove certain isomorphism between spaces, the universality property provides us one starting point: the existence of a morphism.

Proposition B.4. *If the tensor products exists, it is unique up to a isomorphism.*

Proof. Let U, V vector spaces and suppose there are two vector spaces $U \otimes_1 V$ and $U \otimes_2 V$ corresponding to $\otimes_1 : U \times V \to U \otimes_1 V$ and $\otimes_2 : U \times V \to U \otimes_2 V$, respectively, satisfying the conditions of the Definition B.2. Hence, by the universal property there exists a unique linear mapping $\overline{\otimes_2} : U \otimes_1 V \to U \otimes_2 V$ such that the following diagram commutes

$$\begin{tikzcd}
U \times V \arrow[r, "\otimes_1"] \arrow[dr, "\otimes_2"'] & U \otimes_1 V \arrow[d, dashed, "\overline{\otimes_2}"] \\
 & U \otimes_2 V
\end{tikzcd}$$

By the same argument, there exists a unique linear mapping $\overline{\otimes_1} : U \otimes_2 V \to U \otimes_1 V$ such that the following diagram commutes

$$\begin{tikzcd}
U \times V \arrow[r, "\otimes_2"] \arrow[dr, "\otimes_1"'] & U \otimes_2 V \arrow[d, dashed, "\overline{\otimes_1}"] \\
 & U \otimes_1 V
\end{tikzcd}$$

Define

$$\begin{aligned} f_1 &= \overline{\otimes_1} \circ \overline{\otimes_2} : U \otimes_1 V \to U \otimes_1 V \\ f_2 &= \overline{\otimes_2} \circ \overline{\otimes_1} : U \otimes_2 V \to U \otimes_2 V \end{aligned} \tag{B.7}$$

Then we have the following commutative diagrams

$$\begin{tikzcd}
U \times V \arrow[r, "\otimes_1"] \arrow[dr, "\otimes_1"'] & U \otimes_1 V \arrow[d, dashed, "f_1"] \\
 & U \otimes_1 V
\end{tikzcd}
\qquad
\begin{tikzcd}
U \times V \arrow[r, "\otimes_2"] \arrow[dr, "\otimes_2"'] & U \otimes_2 V \arrow[d, dashed, "f_2"] \\
 & U \otimes_2 V
\end{tikzcd}$$

Conversely, the identity mappings $id_1 : U \otimes_1 V \to U \otimes_1 V$, $id_2 : U \otimes_2 V \to U \otimes_2 V$ also make the respective diagrams commute. By uniqueness, $f_1 = id_1$ and $f_2 = id_2$, i.e., $\overline{\otimes_1}$ and $\overline{\otimes_2}$ are inverses of each other. Therefore we conclude that $U \otimes_1 V \simeq U \otimes_2 V$ as desired. □

Proposition B.5. *The following isomorphisms between tensor products of vector spaces hold [2, 3, 6, 8, 9]:*

- **(i)** $U \otimes V \simeq V \otimes U$,
- **(ii)** $U \otimes (V \otimes W) \simeq (U \otimes V) \otimes W$,
- **(iii)** $V \otimes_{\mathbb{K}} \mathbb{K} \simeq V$,
- **(iv)** $U^* \otimes V \simeq \mathcal{L}(U, V)$,
- **(v)** $U^* \otimes V^* \simeq (U \otimes V)^*$.

The tensor product is not commutative although the above item **(i)** establishes that there is an isomorphism between these spaces. The item **(ii)** ensures that we can take the tensor product of an arbitrary number of vector spaces consistently. The item **(iv)** and **(v)** holds only for finite dimensions and it is worth to pointing out that for the item **(iv)**, given $\alpha \in U^*$ and $v \in V$ one can define the following linear mapping [8]

$$\begin{aligned} \otimes : U^* \times V &\to \mathcal{L}(U, V) \\ (\alpha, \mathbf{v}) &\mapsto (\alpha \otimes \mathbf{v}) : U \to V \\ &\qquad\qquad \mathbf{u} \mapsto (\alpha \otimes \mathbf{v})(\mathbf{u}) := \alpha(\mathbf{u})\mathbf{v}. \end{aligned} \tag{B.8}$$

We can also generalise the Definition B.2 for $p \in \mathbb{Z}$ vector spaces over the same field $\mathbb{K}$.

Definition B.6. *Let $V_1, \ldots, V_p$ be $\mathbb{K}$-vector spaces. The **tensor product** (over $\mathbb{K}$) for $V_1, \ldots, V_p$ is the 2-uple $(\otimes, T)$ formed by a vector space T and a p-linear mapping $\otimes : V_1 \times \cdots \times V_p \to T$ such that for any other p-linear mapping $f : V_1 \times \cdots \times V_p \to W$, where W is a vector space, there exists a unique linear mapping $\overline{f} : T \to W$ such that $f = \overline{f} \circ \otimes$ i.e., the following diagram commutes*

$$\begin{tikzcd} V_1 \times \cdots \times V_p \arrow[r, "\otimes"] \arrow[dr, "f"'] & T \arrow[d, dashed, "\overline{f}"] \\ & W \end{tikzcd}$$

The next goal is to prove the existence of the tensor product, some important definitions must be stated first. Let $\mathcal{F} = \{V_i \mid i \in I\}$ be any family of vector spaces over $\mathbb{K}$ [9]:

Definition B.7. *The **direct product** of $\mathcal{F}$ is the vector space*

$$\prod_{i\in I} V_i = \left\{ f : I \to \bigcup_{i\in I} V_i \mid f(i) \in V_i \right\} \tag{B.9}$$

thought of as a subspace of the vector space of all functions from I to $\bigcup V_i$.

Definition B.8. *The **support** of a function $f : I \to \bigcup V_i$ is the set*

$$\operatorname{supp}(f) = \{i \in I \mid f(i) \neq 0\}. \tag{B.10}$$

Thus, a function f has finite support if $f(i) = 0$ for all but finitely many index $i \in I$.

Definition B.9. *The **external direct sum** of the family $\mathcal{F}$ is the vector space*

$$\bigoplus_{i\in I}^{\text{ext}} V_i = \left\{ f : I \to \bigcup_{i\in I} V_i \mid \operatorname{supp}(f) < \infty \right\} \tag{B.11}$$

thought of as a subspace of the vector space of all functions from I to $\bigcup V_i$.

The following result is fundamental to our goal.

Lemma B.10. *Given a set X there exists a $\mathbb{K}$-vector space $F(X)$ such that dim $F(X) = |X|$, i.e., there is a bijective function between X and any basis of $F(X)$.*

Proof. Consider $F(X)$ as

$$F(X) = \bigoplus_{x\in X}^{\text{ext}} \mathbb{K} = \{f : X \to \mathbb{K} \mid \operatorname{supp}(f) < \infty\}. \tag{B.12}$$

In this way, given $f \in F(X)$ we can write f in a unique way as $f = \sum_{x\in X} f(x)\delta_x$ such that $\delta_x : X \to \mathbb{K}$ is defined by

$$\delta_x(y) = \begin{cases} 1_{\mathbb{K}}, \text{ if } x = y, \\ 0_{\mathbb{K}}, \text{ if } x \neq y. \end{cases} \tag{B.13}$$

Since $\operatorname{supp}(\delta_x) < \infty$, $\delta_x \in F(X)$. Also, $f = \sum_{x\in X} f(x)\delta_x < \infty$ since $f(x) \neq 0$ for all but finite $x \in X$. Therefore, $\{\delta_x \mid x \in X\}$ is a basis for $F(X)$ and one can identify X and $\{\delta_x \mid x \in X\}$ through the bijection $x \mapsto \delta_x$, which gives the desired result. □

Theorem B.11. ***(Existence of the Tensor Product)*** *For any family $V_1, \dots, V_p$ of $\mathbb{K}$-vector spaces there exists the tensor product $(\otimes, V)$.*

Proof. Consider the set $V_1 \times \cdots \times V_p$, according to the previous Lemma B.10, there is a $\mathbb{K}$-vector space $F = F(V_1 \times \cdots \times V_p)$ such that $\varphi : V_1 \times \cdots \times V_p \to \mathcal{B}$ is a bijection and $\mathcal{B}$ is some basis for F since $\dim F = |V_1 \times \cdots \times V_p|$. Now, let us consider a vector subspace U of F generated by elements of the form

$$\begin{aligned} &\textbf{(i)}\ \ \varphi(\mathbf{v}_1, \dots, \mathbf{v}_i + \mathbf{u}_i, \dots, \mathbf{v}_p) - \varphi(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_p) - \varphi(\mathbf{v}_1, \dots, \mathbf{u}_i, \dots, \mathbf{v}_p). \\ &\textbf{(ii)}\ \ \varphi(\mathbf{v}_1, \dots, \lambda\mathbf{v}_i, \dots, \mathbf{v}_p) - \lambda\varphi(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_p). \end{aligned} \tag{B.14}$$

Right, let $\otimes : V_1 \times \cdots \times V_p \to F/U = V$ given by $\otimes = \pi \circ \iota \circ \varphi$ s.t $\pi : F \to F/U$ is the quotient mapping and $\iota : \mathcal{B} \hookrightarrow F$ is the inclusion.

$$\begin{tikzcd} V_1 \times \cdots \times V_p \arrow[r, "\varphi"] \arrow[rrr, bend left, "\otimes"] & \mathcal{B} \arrow[r, hook, "\iota"] & F \arrow[r, "\pi"] & F/U = V \end{tikzcd}$$

We claim that $\otimes$ is p-linear. Indeed, because of the item **(i)** written in the expression (B.14), it follows that

$$\begin{aligned} \otimes((\mathbf{v}_1, \dots, \mathbf{v}_i + \mathbf{u}_i, \dots, \mathbf{v}_p) &= (\pi \circ \iota \circ \varphi)(\mathbf{v}_1, \dots, \mathbf{v}_i + \mathbf{u}_i, \dots, \mathbf{v}_p) \\ &= \pi(\varphi(\mathbf{v}_1, \dots, \mathbf{v}_i + \mathbf{u}_i, \dots, \mathbf{v}_p)) \\ &= [\varphi(\mathbf{v}_1, \dots, \mathbf{v}_i + \mathbf{u}_i, \dots, \mathbf{v}_p)] \\ &= [\varphi(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_p)] + [\varphi(\mathbf{v}_1, \dots, \mathbf{u}_i, \dots, \mathbf{v}_p)]. \end{aligned} \tag{B.15}$$

Continuing in this way,

$$\begin{aligned} \otimes(\mathbf{v}_1, \dots, \mathbf{v}_i + \mathbf{u}_i, \dots, \mathbf{v}_p) &= [\varphi(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_p)] + [\varphi(\mathbf{v}_1, \dots, \mathbf{u}_i, \dots, \mathbf{v}_p)] \\ &= \pi(\varphi(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_p)) + \pi(\varphi(\mathbf{v}_1, \dots, \mathbf{u}_i, \dots, \mathbf{v}_p)) \\ &= (\pi \circ \iota \circ \varphi)(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_p)) + (\pi \circ \iota \circ \varphi)(\mathbf{v}_1, \dots, \mathbf{u}_i, \dots, \mathbf{v}_p)) \\ &= \otimes(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_p)) + \otimes(\mathbf{v}_1, \dots, \mathbf{u}_i, \dots, \mathbf{v}_p)). \end{aligned} \tag{B.16}$$

Analogously, by the item **(ii)** written in the Eq. (B.14) it also holds that for $\lambda \in \mathbb{K}$

$$\otimes(\mathbf{v}_1, \dots, \lambda\mathbf{v}_i, \dots, \mathbf{v}_p) = \lambda \otimes (\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_p). \tag{B.17}$$

Which proves the p-linearity of $\otimes$. We claim that universality holds for $(\otimes, V)$. Indeed, let W be an arbitrary vector space over $\mathbb{K}$ and $\psi : V_1 \times \cdots \times V_p$ an arbitrary p-linear mapping.

$$\begin{tikzcd}
V_1 \times \cdots \times V_p \arrow[r, "\varphi"] \arrow[rrr, bend left, "\otimes"] \arrow[d, "\psi"'] & \mathcal{B} \arrow[r, hook, "i"] & F \arrow[r, "\pi"] & F/U = V \\
W & & &
\end{tikzcd}$$

Since φ is a bijection one can consider φ^{-1}. In addition, since $\mathcal{B}$ is a basis for F and $\psi \circ \varphi^{-1} : \mathcal{B} \to W$ is a mapping defined by basis, $\psi \circ \varphi^{-1}$ can be extended uniquely to a linear mapping $\Psi_1 : F \to W$ such that $\Psi_1 \circ \iota = \psi \circ \varphi^{-1}$, therefore, $\psi = \Psi_1 \circ \iota \circ \varphi$

$$\begin{tikzcd}
V_1 \times \cdots \times V_p \arrow[r, "\varphi"] \arrow[rrr, bend left, "\otimes"] \arrow[d, "\psi"'] & \mathcal{B} \arrow[r, hook, "\iota"] & F \arrow[r, "\pi"] \arrow[dll, dashed, "\Psi_1"'] & F/U = V \\
W & & &
\end{tikzcd}$$

It is of our interest to investigate how Ψ_1 acts on the vector space $U \subset F$. We claim that $U \subset \ker\Psi_1$. It is sufficient to ascertain $\Psi_1(x)$ for x as a generator of U written in the Eq. (B.14). For instance, let the generator $x \in U$ be as the item **(ii)**

$$x = \varphi(\mathbf{v}_1, \ldots, \lambda\mathbf{v}_i, \ldots, \mathbf{v}_p) - \lambda\varphi(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p). \tag{B.18}$$

Therefore by the linearity of Ψ_1 and the p-linearity of ψ, we have

$$\begin{aligned}
\Psi_1(x) &= \Psi_1(\varphi(\mathbf{v}_1, \ldots, \lambda\mathbf{v}_i, \ldots, \mathbf{v}_p) - \lambda\varphi(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p)) \\
&= \Psi_1(\varphi(\mathbf{v}_1, \ldots, \lambda\mathbf{v}_i, \ldots, \mathbf{v}_p) - \lambda\Psi_1(\varphi(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p)) \\
&= \Psi_1(\iota \circ \varphi(\mathbf{v}_1, \ldots, \lambda\mathbf{v}_i, \ldots, \mathbf{v}_p) - \lambda\Psi_1(\iota \circ \varphi(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p)) \\
&= \psi(\mathbf{v}_1, \ldots, \lambda\mathbf{v}_i, \ldots, \mathbf{v}_p) - \lambda\psi(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p)) \\
&= \lambda\psi(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p) - \lambda\psi(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p)) = 0.
\end{aligned} \tag{B.19}$$

Analogously, for $x \in U$ as the item **(i)** in the Eq. (B.14) we have that $\Psi_1(x) = 0$. Hence, by concluding that for any generator x of U, $\Psi(x) = 0$, we have that $U \subset \ker(f)$. By the Example B.3, it follows that there exists a unique linear mapping $\Psi_2 : F/U \to W$ such that $\Psi_2 \circ \pi = \Psi_1$

$$\begin{tikzcd}
V_1 \times \cdots \times V_p \arrow[r, "\varphi"] \arrow[rrr, bend left, "\otimes"] \arrow[d, "\psi"'] & \mathcal{B} \arrow[r, hook, "\iota"] & F \arrow[r, "\pi"] \arrow[dll, dashed, "\Psi_1"'] & F/U = V \arrow[dlll, dashed, "\Psi_2"] \\
W & & &
\end{tikzcd}$$

Therefore, we have that

$$\Psi_2 \circ \otimes = \Psi_2 \circ (\pi \circ \iota \circ \varphi) = (\Psi_2 \circ \pi) \circ \iota \circ \varphi = \Psi_1 \circ \iota \circ \varphi = \psi \tag{B.20}$$

Right, we claim that Ψ_2 is unique. Indeed, suppose that the diagram commutes with respect to another linear mapping $f : F/U \to W$

$$\begin{tikzcd}
V_1 \times \cdots \times V_p \arrow[r, "\otimes"] \arrow[d, "\psi"'] & F/U = V \arrow[dl, dashed, "f"] \\
W &
\end{tikzcd}$$

By Eq. (B.20), we have that

$$f \circ \otimes = \psi = \Psi_2 \circ \otimes \tag{B.21}$$

Which means that f and Ψ_2 coincides on the elements of the form

$$\begin{aligned} \otimes(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p) &= \pi \circ \iota \circ \varphi(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p) \\ &= [\varphi(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p)]. \end{aligned} \tag{B.22}$$

We have that $[\varphi(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_p)]$ is the equivalent class of the basis elements of F, namely, the generator set of F/U. Hence, f and Ψ_2 coincides on entire space F/U and the uniqueness holds. We conclude that $(\otimes, V)$ with $V = F/U$, is the tensor product of for $V_1, \ldots, V_p$ by the Definition B.6. □

It is worth to mention that one can also write the tensor product $(\otimes, V)$ of $V_1, \ldots, V_p$ as $V_1 \otimes \cdots \otimes V_p$. It follows that $\mathcal{L}(V_1, \ldots, V_p; W) \simeq \mathcal{L}(V_1 \otimes \cdots \otimes V_p, W)$ by the universality. Furthermore, if the vector spaces are all equal $V_i = V$ we denote the tensor product as [8]

$$T_q(V) = \overbrace{V \otimes \cdots \otimes V}^{q \text{ factors}} = V^{\otimes q} \tag{B.23}$$

Analogously, the tensor product of p dual spaces V^* is written as

$$T^p(V) = \overbrace{V^* \otimes \cdots \otimes V^*}^{p \text{ factors}} = (V^*)^{\otimes p} \tag{B.24}$$

In addition,

$$\begin{aligned} T^p{}_q(V) &= (V^*)^{\otimes p} \otimes V^{\otimes q} \\ T_q{}^p(V) &= V^{\otimes q} \otimes (V^*)^{\otimes p} \end{aligned} \tag{B.25}$$

We adopt the convention that $T^p_q(V)$ refers to the space $T^p{}_q(V)$. A basis for the vector space $T^p_q(V)$ is given by the set of tensor products [7]

$$\{\mathbf{e}^{\mu_1} \otimes \mathbf{e}^{\mu_2} \otimes \cdots \otimes \mathbf{e}^{\mu_p} \otimes \mathbf{e}_{\nu_1} \otimes \mathbf{e}_{\nu_2} \otimes \cdots \otimes \mathbf{e}_{\nu_q}\}, \tag{B.26}$$

An arbitrary element $T \in T^p_q(V)$ can be written as

$$T = T^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p} \mathbf{e}^{\mu_1} \otimes \mathbf{e}^{\mu_2} \otimes \cdots \otimes \mathbf{e}^{\mu_p} \otimes \mathbf{e}_{\nu_1} \otimes \mathbf{e}_{\nu_2} \otimes \cdots \otimes \mathbf{e}_{\nu_q}, \tag{B.27}$$

such that

$$T^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p} = T(\mathbf{e}_{\mu_1}, \mathbf{e}_{\mu_2}, \ldots, \mathbf{e}_{\mu_p}, \mathbf{e}^{\nu_1}, \mathbf{e}^{\nu_2}, \ldots, \mathbf{e}^{\nu_q}). \tag{B.28}$$

The multilinear functionals $T \in T^p_q(V)$ are called *tensors of type (p,q)*. The quantities $T^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p}$ are the components of the tensor T in the given basis. The tensors of type $(p,0)$ are called *covariant tensors* as well as tensors of type $(0,q)$ are called *contravariant tensors.*

Example 2.12 ▹ Type of tensors.

We assume that tensors of type $(0,0)$ are scalars, tensors of type $(0,1)$ are vectors and tensors of type $(1,0)$ are covectors, in other words, $T^0_0(V) = \mathbb{K}$, $T^0_1(V) = V$, $T^1_0(V) = V^*$ and more generally [8]

$$\begin{aligned} T^q_0(V) &= \mathcal{L}(V, \ldots, V; \mathbb{K}), \\ T^q_1(V) &= \mathcal{L}(V, \ldots, V; V). \end{aligned} \tag{B.29}$$

In particular tensors of type $(0,2)$ are bilinear mappings and tensors of type $(1,1)$ are linear mappings. ◃

B.2. Tensor Algebra

We are now in a position to introduce the tensor algebra. Given two tensors T and S of type (p,q) we define their sum as the tensors $T+S$ of type (p,q) in terms of their components by [7]

$$(T+S)^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p} = T^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p} + S^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p}. \tag{B.30}$$

If T is a tensor of type (p,q) and S is a tensor of type (r,s), the tensor product $T\otimes S$, which is a tensor of type $(p+r,q+s)$, is defined in terms of their components as

$$(T\otimes S)^{\nu_1\nu_2\cdots\nu_q\rho_1\rho_2\cdots\rho_s}_{\mu_1\mu_2\cdots\mu_p\sigma_1\sigma_2\cdots\sigma_r} = T^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p} S^{\rho_1\rho_2\cdots\rho_s}_{\sigma_1\sigma_2\cdots\sigma_r} \tag{B.31}$$

The product $\otimes$ is distributive with respect to sum and is associative but is not commutative.

Definition B.13. *The direct sum of all vector spaces $T^p_q(V)$ endowed with the operations of sum and the tensor product is called the* ***tensor algebra*** *associated to the vector space V*

The tensor algebra is a graded algebra. In the general case, the grading is given by $G=\mathbb{Z}\times\mathbb{Z}$ and it is positive. Two cases are particularly important: the algebra of covariant and contravariant tensors. The algebra of the covariant tensors, of type $(p,0)$, is denoted by

$$T^*(V) = \bigoplus_{p=0}^{\infty} T^p(V) \tag{B.32}$$

whereas

$$T(V) = \bigoplus_{p=0}^{\infty} T^p(V) \tag{B.33}$$

is the covariant tensor algebra of $(0,q)$ tensors.

The tensor algebra is fundamental because many other algebras arise as quotient algebras of T(V). For instance, the exterior algebra and the Clifford algebra are introduced in this work as a quotient of tensor algebra.

C EXTERIOR ALGEBRA

In the study of alternating multilinear mappings and their ramifications, exterior algebra stands out. Introduced by Hermann Grassmann in 1844, from a geometric point of view, exterior algebra is an algebraic construction of multidimensional vectors in which they have the meaning of points, oriented line segment, oriented plane fragment, oriented volume fragment, and so on. This algebra has a great richness in its structure and its importance lies not only in its applications in the construction of physical theories, algebraic topology, differential forms but also in algebra, whose consequences will be explored since the exterior algebra provides the multivector structure for Clifford algebra and it is a Clifford algebra itself. Therefore in this appendix, we will introduce the general concepts about exterior algebra. The starting point will be the alternator operator and from such operator, we will define the exterior algebra elements, the underlying vector space, the exterior product, and some properties, operations, that it is used in this entire work.

C.1. Permutations and the Alternator

Let $\{1,2,\ldots,p\}$ be a set of p elements. A **permutation** is a bijective function $\sigma : \{1,2,\ldots,p\} \to \{1,2,\ldots,p\}$ represented by the cycle

$$\begin{pmatrix} 1 & 2 & \cdots & p \\ \sigma(1) & \sigma(2) & \cdots & \sigma(p) \end{pmatrix} \tag{C.1}$$

A permutation σ such that $\sigma(k) = k$ for all $k \neq i, k \neq j$ and $\sigma(i) = j, \sigma(j) = i$, is called a *transposition* and can be denoted by (ij). A permutation of n elements is said to be *even* or *odd* if the permutation can be written respectively as an even or an odd number of transpositions. Therefore the *sign* $\varepsilon(\sigma)$ of a permutation σ is defined to be $\varepsilon(\sigma) = +1$ if the permutation is even and $\varepsilon(\sigma) = -1$ if the permutation is odd. The composition of two permutations is also a permutation and the set of all permutations form a group, namely, *symmetric group* S_p and it has $p!$ elements.

Right, let $X_1 \otimes X_2 \otimes \cdots \otimes X_p$ be either a contravariant or covariant tensor such that where X denotes respectively either a vector or a covector, and the indexes enumerate such elements. We define the **operator alternator** denoted by Alt as follows

$$\mathrm{Alt}(X_1 \otimes X_2 \otimes \cdots \otimes X_p) = \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon(\sigma) X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(p)} \tag{C.2}$$

The operator Alt is a projection operator ($\mathrm{Alt}^2 = \mathrm{Alt}$). The alternator is the starting point in the construction of the exterior algebra.

Example 3.1 ▹ S_3

Let us consider the symmetric group S_3 given by the permutations of 3 elements. Those permutations are represented by

$$\begin{aligned} &\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \leftrightarrow (1)(2)(3), \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \leftrightarrow (12)(13), \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \leftrightarrow (13)(12), \\ &\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \leftrightarrow (13)(2), \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \leftrightarrow (12)(3), \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \leftrightarrow (23)(1). \end{aligned} \tag{C.3}$$

Therefore we can see that the elements in the first row of the above equation are all even permutations whereas the ones in the second row are all odd. For either a covariant or contravariant tensor $X_1 \otimes X_2 \otimes X_3$ the alternator is given by

$$\begin{aligned} \mathrm{Alt}(X_1 \otimes X_2 \otimes X_3) = \frac{1}{6}(&X_1 \otimes X_2 \otimes X_3 + X_2 \otimes X_3 \otimes X_1 + X_3 \otimes X_1 \otimes X_2 \\ &- X_3 \otimes X_2 \otimes X_1 - X_2 \otimes X_1 \otimes X_3 - X_1 \otimes X_3 \otimes X_2). \end{aligned} \tag{C.4}$$ ◃

Another way to represent the action of the alternation is when a covariant tensor is taken into account. Those objects are multilinear functionals acting on vectors. Without loss of generality, let $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_p$ be a covariant tensor, it reads:

$$\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_p(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p) = \alpha_1(\mathbf{v}_1) \otimes \alpha_2(\mathbf{v}_2) \otimes \cdots \otimes \alpha_p(\mathbf{v}_p). \tag{C.5}$$

The action of Alt on a contravariant tensor is defined by

$$\mathrm{Alt}(\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_p)(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p) = \frac{1}{p!} \begin{vmatrix} \alpha_1(\mathbf{v}_1) & \alpha_1(\mathbf{v}_2) & \cdots & \alpha_1(\mathbf{v}_p) \\ \alpha_2(\mathbf{v}_1) & \alpha_2(\mathbf{v}_2) & \cdots & \alpha_2(\mathbf{v}_p) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_p(\mathbf{v}_1) & \alpha_p(\mathbf{v}_2) & \cdots & \alpha_p(\mathbf{v}_p) \end{vmatrix}. \tag{C.6}$$

Such that at the right-hand side of the above equation there is the determinant of the associated matrix. This result follows from the definition of the determinant, we have that if A is the matrix of order p with entries A_{ij} then the determinant det A is given by

$$\det A = \sum_{\sigma \in S_p} \varepsilon(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{p\sigma(p)}. \tag{C.7}$$

C.2. p-vectors and p-covectors

Once the alternator has been presented, we are now in a position to introduce some elements that can be constructed from it.

Definition C.2. *A* ***p-vector*** *is an alternating contravariant tensor of order p denoted by* $A_{[p]}$*. A* ***p-covector*** *is an alternating covariant tensor of order p denoted by* $\Psi^{[p]}$*. The p-vector and the p-covector are characterised by*

$$\begin{aligned} A_{[p]} &= \mathrm{Alt}(A_{[p]}), \\ \Psi^{[p]} &= \mathrm{Alt}(\Psi^{[p]}). \end{aligned} \tag{C.8}$$

Let V be a real vector space, the space of p-vectors and p-covectors are denoted respectively by $\bigwedge_p(V)$ and $\bigwedge^p(V)$ in such way that

$$\bigwedge_0(V) = \bigwedge^0(V) = \mathbb{R}, \quad \bigwedge_1(V) = V, \quad \bigwedge^1(V) = V^*. \tag{C.9}$$

Both 0-vectors and 0-covectors are scalars, 1-vector is a synonym of vector, as well as 1-covector is a synonym of covector. It is also common to call a 2-vector a bivector, 3-vector a trivector, and so on. The important point to note here is the same once one construction involving p-vectors is accomplished, the same reasoning applies to the construction concerning p-covectors. Thus, just the case involving p-vectors shall be considered subsequently. Our next concern will be to introduce a product between those elements.

C.3. Exterior Product

Let $A_{[p]}$ be a p-vector and $B_{[q]}$ be a q-vector, the result of $A_{[p]} \otimes B_{[q]}$ is a contravariant tensor of order $p+q$ however it is not alternating. Meanwhile, $\mathrm{Alt}(A_{[p]} \otimes B_{[q]})$ is an alternating contravariant tensor of order $p+q$, namely, a $(p+q)$-vector. That motivates the following definition.

Definition C.3. *Let V be a vector space, $A_{[p]} \in \bigwedge_p(V)$ a p-vector and $B_{[q]} \in \bigwedge_q(V)$ a q-vector. The* ***exterior product*** $\wedge : \bigwedge_p(V) \times \bigwedge_q(V) \to \bigwedge_{p+q}(V)$ *is defined as*

$$A_{[p]} \wedge B_{[q]} = \mathrm{Alt}(A_{[p]} \otimes B_{[q]}). \tag{C.10}$$

Since the tensor product is associative and bilinear, the exterior product inherits the associativity and bilinearity [7]. That is: for $A_{[p]} \in \bigwedge_p(V), B_{[q]} \in \bigwedge_q(V), C_{[r]} \in \bigwedge_r(V), aA_{[p]} \in \bigwedge_0(V)$ it holds

(i) $(A_{[p]} \wedge B_{[q]}) \wedge C_{[r]} = A_{[p]} \wedge (B_{[q]} \wedge C_{[r]})$,

(ii) $A_{[p]} \wedge (B_{[q]} + C_{[r]}) = A_{[p]} \wedge B_{[q]} + A_{[p]} \wedge C_{[r]}$,

(iii) $a \wedge A_{[p]} = aA_{[p]}$,

(iv) $A_{[p]} \wedge B_{[q]} = (-1)^{pq} B_{[q]} \wedge A_{[p]}$.

In order to derive the item **(iv)**, it is of our interest to study the case involving the exterior product between two vectors. It follows from Definition C.3 that

$$\mathbf{v} \wedge \mathbf{u} = \frac{1}{2}(\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}) \tag{C.11}$$

which gives

$$\mathbf{v} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{v}, \tag{C.12}$$

that is, the exterior product is anti-commutative. In particular $\mathbf{v} \wedge \mathbf{v} = 0$. For the general case, a p-vector $A_{[p]}$ and a q-vector $B_{[q]}$ can be written in the form

$$A_{[p]} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p, \quad B_{[q]} = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_p. \tag{C.13}$$

That is due to the fact that the exterior product is bilinear and associative. Hence, the exterior product $A_{[p]} \wedge B_{[q]}$ reads

$$A_{[p]} \wedge B_{[q]} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p \wedge \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_p. \tag{C.14}$$

In order to interchange the vectors involved in the exterior products, we conclude from Eq. (C.13) that

$$\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p \wedge \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_q = (-1)^{pq} \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_p \wedge \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p \tag{C.15}$$

which gives the **(iv)** property of the exterior product $A_{[p]} \wedge B_{[q]} = (-1)^{pq} B_{[q]} \wedge A_{[p]}$.

We emphasise that a p-vector that can be written as the exterior product of a p number of vectors as in Eq. (C.13), is called a *simple* p-vector. Considering vector spaces V such that $\dim V \leq 3$, every p-vector is simple for higher dimensions not all p-vectors are simple [7].

Regarding the space of the p-vectors, an important point about it is with respect to its basis. Let V be a vector space with basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, a basis for each one of the spaces $\bigwedge_p(V)$ can be constructed from the basis of V. Let us first examine the space $\bigwedge_2(V)$ and the exterior products $\mathbf{e}_i \wedge \mathbf{e}_j$. Since the exterior product is anti-commutative, the linearly independent set of bivectors is provided by

$$\begin{aligned} \mathbf{e}_1 \wedge \mathbf{e}_2,\ \mathbf{e}_1 \wedge \mathbf{e}_3,\ \mathbf{e}_1 \wedge \mathbf{e}_4,\ \cdots,\ \mathbf{e}_1 \wedge \mathbf{e}_n, \\ \mathbf{e}_2 \wedge \mathbf{e}_3,\ \mathbf{e}_2 \wedge \mathbf{e}_4,\ \cdots,\ \mathbf{e}_2 \wedge \mathbf{e}_n, \\ \vdots \\ \mathbf{e}_{n-1} \wedge \mathbf{e}_n. \end{aligned} \tag{C.16}$$

Therefore, the dimension of $\bigwedge_2(V)$ is the number of possible combinations of $n = \dim V$ vectors taken 2 at a time. For the general case,

$$\dim \bigwedge_p (V) = \binom{n}{p} = \frac{n!}{(n-p)!p!}. \tag{C.17}$$

An arbitrary element $A_{[p]} \in \bigwedge_p(V)$ can be written as

$$\begin{aligned} A_{[p]} &= \frac{1}{p!} \sum_{\mu_1\mu_2\cdots\mu_p} A^{\mu_1\mu_2\cdots\mu_p} \mathbf{e}_{\mu_1} \wedge \mathbf{e}_{\mu_2} \wedge \cdots \wedge \mathbf{e}_{\mu_p} \\ &= \sum_{\mu_1<\mu_2<\cdots<\mu_p} A^{\mu_1\mu_2\cdots\mu_p} \mathbf{e}_{\mu_1} \wedge \mathbf{e}_{\mu_2} \wedge \cdots \wedge \mathbf{e}_{\mu_p} \end{aligned} \tag{C.18}$$

Regarding V a vector space with $\dim V = n$, one may ask now how many spaces $\bigwedge_p(V)$ can be constructed from V taking into account the dimension of V. We begin with a general result about the exterior product

Proposition C.4. *If $p > n = \dim V$, the exterior product of p vectors is null.*

Proof. Let us consider the exterior product of $n+1$ vectors, $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n \wedge \mathbf{v}_{n+1}$. Since $\dim V = n$, the $n+1$ given vectors are necessarily linearly dependent and we can write one of those vectors as a linear combination of the others. There is no loss of generality in assuming that $\mathbf{v}_{n+1} = \sum_{i=1}^{n} a^i \mathbf{v}_i$. On account of the anticommutativity and the fact that $\mathbf{v}_i \wedge \mathbf{v}_i = 0$ we have that:

$$\begin{aligned}\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n \wedge \mathbf{v}_{n+1} &= \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n \wedge \left(a^1\mathbf{v}_1 + a^2\mathbf{v}_2 + \cdots + a^n\mathbf{v}_n\right) \\ &= (-1)^{n-1}a^1\mathbf{v}_1 \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n \\ &\quad + (-1)^{n-2}a^2\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n \\ &\quad + \cdots + a^n\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n \wedge \mathbf{v}_n \\ &= 0\end{aligned} \tag{C.19}$$

which completes the proof. □

More generally, we have thus proved that

$\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p = 0 \Longleftrightarrow \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

That discussion clarifies that it does not exists the vector space $\bigwedge_p(V)$ if $p > n$. Therefore the spaces that can be constructed are

$$\bigwedge_0(V), \bigwedge_1(V), \bigwedge_2(V), \ldots, \bigwedge_{n-1}(V), \bigwedge_p(V). \tag{C.20}$$

Such that

$$\dim \bigwedge_p(V) = \binom{n}{p} = \frac{n!}{(n-p)!p!} = \binom{n}{n-p} = \dim \bigwedge_{n-p}(V) \tag{C.21}$$

Although the spaces $\bigwedge_p(V)$ and $\bigwedge_{n-p}(V)$ are isomorphic, there is not a natural isomorphism between them. However, by considering additional structures on the vector space V , it is possible to construct an isomorphism called Hodge isomorphism which can be found in detail in the reference [7].

C.4. Exterior Algebra

We regard the exterior product $\wedge$ as being defined on $\bigwedge_p(V) \times \bigwedge_q(V) \to \bigwedge_{p+q}(V)$. Let us consider the vector space $\bigwedge(V)$ defined by the direct sum of the vector spaces $\bigwedge_p(V)$, $(p = 0, 1, 2, \ldots, n)$:

$$\bigwedge(V) = \bigwedge_0(V) \oplus \bigwedge_1(V) \oplus \bigwedge_2(V) \oplus \cdots \oplus \bigwedge_n(V) = \bigoplus_{p=0}^{n} \bigwedge_p(V). \tag{C.22}$$

The space $\bigwedge(V)$ is thereby closed by the exterior product, that is, $\bigwedge(V) \times \bigwedge(V) \to \bigwedge(V)$.

Definition C.5. *The pair* $(\bigwedge(V), \wedge)$ *is named* ***exterior algebra*** *associated to the vector space* V.

The elements of the space $\bigwedge(V)$ are called *multivectors*. An arbitrary multivector $A \in \bigwedge(V)$ is written as

$$\begin{aligned} A = \underbrace{a}_{scalar} + \underbrace{v^i \mathbf{e}_i}_{vector} + \underbrace{F^{ij}\mathbf{e}_i \wedge \mathbf{e}_j}_{2-vector} \\ + \underbrace{T^{ijk}\mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k}_{3-vector} + \cdots + \underbrace{p\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n}_{n-vector} \in \bigwedge(V). \end{aligned} \tag{C.23}$$

According to our previous discussion, it follows that the dimension of $\bigwedge(V)$ is given by

$$\dim \bigwedge(V) = \sum_{p=0}^{n} \dim \bigwedge_{p}(V) = \sum_{p=0}^{n} \binom{n}{p} = 2^n. \tag{C.24}$$

Example 3.6 ▹ The exterior algebra $\bigwedge(\mathbb{R}^3)$

Let us consider the 3-dimensional euclidean $V = \mathbb{R}^3$ with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. These basis elements satisfies the following relation:

$$\begin{aligned} \mathbf{e}_i \wedge \mathbf{e}_i &= 0, && (\text{for } i = 1,2,3). \\ \mathbf{e}_i \wedge \mathbf{e}_j &= -\mathbf{e}_j \wedge \mathbf{e}_i, && (\text{for } i \neq j). \end{aligned} \tag{C.25}$$

The exterior algebra is the space

$$\bigwedge(\mathbb{R}^3) = \bigoplus_{p=0}^{3} \bigwedge^{p}(\mathbb{R}^3). \tag{C.26}$$

Let us consider the subspace of the bivectors $\bigwedge^2(\mathbb{R}^3)$ with basis $\{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3\}$. In a geometric point of view, each element $\mathbf{e}_i \wedge \mathbf{e}_j$ represents an oriented plane fragment generated by two vectors as represented in the following figure

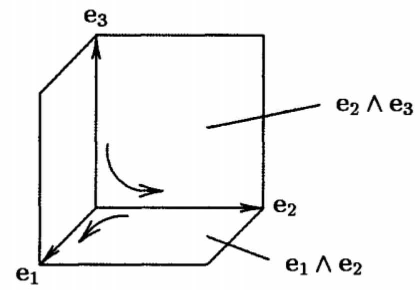


Figure C.1.: The geometry of the elements $\{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3\}$. Source: reference [4].

The exterior product between two vectors $a, b \in \bigwedge^1(\mathbb{R}^3)$ is a bivector and can be computed through the determinant of the following matrix:

$$\begin{aligned}\mathbf{a} \wedge \mathbf{b} &= \begin{vmatrix} \mathbf{e}_2 \wedge \mathbf{e}_3 & \mathbf{e}_3 \wedge \mathbf{e}_1 & \mathbf{e}_1 \wedge \mathbf{e}_2 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2 b_3 - a_3 b_2)\mathbf{e}_2 \wedge \mathbf{e}_3 + (a_3 b_1 - a_1 b_3)\mathbf{e}_3 \wedge \mathbf{e}_1 + (a_1 b_2 - a_2 b_1)\mathbf{e}_1 \wedge \mathbf{e}_2 \\ &= B_{23}\mathbf{e}_2 \wedge \mathbf{e}_3 + B_{31}\mathbf{e}_3 \wedge \mathbf{e}_1 + B_{12}\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{B}. \triangleleft \end{aligned} \tag{C.27}$$

C.5. Exterior Algebra as a Quotient of Tensor Algebra

Our next discussion is devoted to an explicit construction of the exterior algebra as quotient of tensor algebra. Besides being an interesting mathematical construction itself, an analogous approach is used for the Clifford algebras.

Definition C.7. *Let $\mathcal{A}$ be an algebra. A set $I_L \subset \mathcal{A}$ is said to be a* ***left ideal*** *of $\mathcal{A}$ if $\forall\, a \in \mathcal{A}, \forall\, x \in I_L, ax \in I_L$. Analogously, $I_R \subset \mathcal{A}$ is said to be a* ***right ideal*** *of $\mathcal{A}$ if $\forall\, a \in \mathcal{A}, \forall\, x \in I_L, xa \in I_R$. The set $\mathcal{I} \subset \mathcal{A}$ is said to be a* ***two-sided ideal*** *or simply an* ***ideal*** *if $\forall\, a, b \in \mathcal{A}, \forall\, x \in \mathcal{I}, axb \in \mathcal{I}$*

Let $\mathcal{A}$ be an algebra and write A as sum of spaces $\mathcal{A} = \mathcal{B} + \mathcal{C}$. Given $a, b \in \mathcal{A}$ following equivalence relation can be defined:

$$a \sim b \Longleftrightarrow a = b + x, \qquad x \in \mathcal{C}. \tag{C.28}$$

The set of the equivalence classes $A/\sim$ has a natural vector space structure with sum and multiplication by scalar defined by

$$\begin{aligned}[][a] + [b] &= [a + b], \\ \lambda[a] &= [\lambda a].\end{aligned} \tag{C.29}$$

To $A/\sim$ be an algebra, a natural way to define the product between equivalence classes is

$$[a][b] = [ab] \tag{C.30}$$

Since we know by definition that $[a] = [a+x], [b] = [b+y], x, y \in \mathcal{C}$, it follows that

$$[a][b] = [a+x][b+y] = [(a+x)(b+y)] = [ab+ay+xb+xy]. \tag{C.31}$$

The last two equations result in

$$[ab] = [ab+ay+xb+xy]. \tag{C.32}$$

That means that in order to $A/\sim$ be an algebra $ay+xb+xy$ must be an element of $\mathcal{C}$ which only holds if $\mathcal{C}$ is a two-sided ideal. In that case, $A/\sim$ is named the quotient algebra of $\mathcal{A}$ by $\mathcal{C}$, denoted by $\mathcal{A}/\mathcal{C}$.

Let $T(V)$ be the algebra of the contravariant tensors. Consider the ideal I of $T(V)$ generated by elements $\mathbf{v}\otimes\mathbf{v}, \mathbf{v} \in V$. The elements of I consists of the sums

$$\sum_i A_i \otimes \mathbf{v}_i \otimes \mathbf{v}_i \otimes B_i, \tag{C.33}$$

where $\mathbf{v}_i \in V$ and $A_i, B_i \in T(V)$. We can notice that

$$\mathbf{v}\otimes\mathbf{u} + \mathbf{u}\otimes\mathbf{v} = (\mathbf{v}+\mathbf{u})\otimes(\mathbf{v}+\mathbf{u}) - \mathbf{v}\otimes\mathbf{v} - \mathbf{u}\otimes\mathbf{u}. \tag{C.34}$$

That is, we can also consider that the ideal $\mathcal{I}$ is generated by the elements $\mathbf{v}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{v}$ where $\mathbf{v}, \mathbf{u} \in V$. One important point to note here lies in the fact that

$$\mathbf{v}\otimes\mathbf{u} + \mathbf{u}\otimes\mathbf{v} \in \ker \mathrm{Alt} \tag{C.35}$$

and $\ker \mathrm{Alt} = \mathcal{I}$. Our next claim is that the exterior algebra is isomorphic to the quotient algebra $T(V)/\mathcal{I}$. The respective equivalence relation is given by

$$A \sim B \Longleftrightarrow A = B + x, \qquad x \in \mathcal{I}. \tag{C.36}$$

Moreover, the product between them is denoted by

$$[A] \wedge [B] = [A \otimes B]. \tag{C.37}$$

For $\mathbf{v}, \mathbf{u} \in V$ we have that

$$\mathbf{v}\otimes\mathbf{u}=\frac{1}{2}(\mathbf{v}\otimes\mathbf{u}-\mathbf{u}\otimes\mathbf{v})+\frac{1}{2}(\mathbf{v}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{v}) \tag{C.38}$$

such that $\frac{1}{2}(\mathbf{v}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{v})\in\mathcal{I}$. Therefore,

$$[\mathbf{v}]\wedge[\mathbf{u}]=[\mathbf{v}\otimes\mathbf{u}]=[\frac{1}{2}(\mathbf{v}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{v})]=[\mathrm{Alt}(\mathbf{v}\otimes\mathbf{u})]. \tag{C.39}$$

For the general case, the above result is extended as

$$\mathbf{v}_1\otimes\cdots\otimes\mathbf{v}_p\sim\mathrm{Alt}(\mathbf{v}_1\otimes\cdots\otimes\mathbf{v}_p)=\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_p \tag{C.40}$$

which establishes the desired isomorphism

$$\bigwedge(V)\simeq T(V)/\mathcal{I}. \tag{C.41}$$

C.6. Contraction

Let $A_{[p]}$ be a p-vector and α a covector. The final task in this appendix is to present an operation that action on $A_{[p]}\in\bigwedge_p(V)$ and gives an element of $\bigwedge_{p-1}(V)$, i.e., a $(p-1)$-vector. Such operation is very important and it is used to define the Clifford product on Clifford algebras.

Definition C.8. *The* ***left contraction*** *of a p-vector $A_{[p]}$ by a covector α, denoted by $\alpha\rfloor$, is defined as*

$$\begin{aligned}\rfloor:\overset{1}{\bigwedge}(V)\times\bigwedge_p(V)&\longrightarrow\bigwedge_{p-1}(V)\\(\alpha,A_{[p]})&\longmapsto(\alpha\rfloor A_{[p]})(\alpha_1,\ldots,\alpha_{p-1})=pA_{[p]}(\alpha,\alpha_1,\ldots,\alpha_{p-1}).\end{aligned} \tag{C.42}$$

such that $(\alpha_1,\ldots,\alpha_{p-1})$ stands for arbitrary covectors.

On the right-hand side of the above expression, for $A_{[p]}=\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_p$ it means that

$$(\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_p)(\alpha,\alpha_1,\ldots,\alpha_{p-1})=\frac{1}{p!}\sum_{\sigma\in Sp}\varepsilon(\sigma)\alpha(\mathbf{v}_{\sigma(1)})\alpha_1(\mathbf{v}_{\sigma(2)})\cdots\alpha_{p-1}(\mathbf{v}_{\sigma(p)}). \tag{C.43}$$

It follows immediately that for $p=1$

$$\alpha\rfloor\mathbf{v}=\alpha(\mathbf{v}). \tag{C.44}$$

For an element of $\bigwedge_0(V)$ it is assumed that $\alpha\rfloor 1=0$. Our next concern will be the contraction of a 2-vector $\mathbf{v}\wedge\mathbf{u}$. It reads:

$$
\begin{aligned}
(\alpha\rfloor(\mathbf{v}\wedge\mathbf{u}))(\beta) &= 2(\mathbf{v}\wedge\mathbf{u})(\alpha,\beta)\\
&= \alpha(\mathbf{v})\beta(\mathbf{u})-\alpha(\mathbf{u})\beta(\mathbf{v})\\
&= (\alpha(\mathbf{v})\mathbf{u}-\alpha(\mathbf{u})\mathbf{v})(\beta)\\
&= ((\alpha\rfloor\mathbf{v})\mathbf{u}-(\alpha\rfloor\mathbf{u})\mathbf{v})(\beta).
\end{aligned}
\tag{C.45}
$$

Since β is arbitrary, we conclude that

$$
\alpha\rfloor(\mathbf{v}\wedge\mathbf{u}) = ((\alpha\rfloor\mathbf{v})\mathbf{u}-(\alpha\rfloor\mathbf{u})\mathbf{v}) = (\alpha(\mathbf{v})\mathbf{u}-\alpha(\mathbf{u})\mathbf{v}). \tag{C.46}
$$

We proceed to develop the contraction for a 3-vector $\mathbf{v}\wedge\mathbf{u}\wedge\mathbf{w}$ using the above results. It follows that:

$$
\begin{aligned}
(\alpha\rfloor(\mathbf{v}\wedge\mathbf{u}\wedge\mathbf{w})(\beta,\gamma) &= 3(\mathbf{v}\wedge\mathbf{u}\wedge\mathbf{w})(\alpha,\beta,\gamma)\\
&= 3\frac{1}{3!}[\alpha(\mathbf{v})\beta(\mathbf{u})\gamma(\mathbf{w}))+\beta(\mathbf{v})\gamma(\mathbf{u})\alpha(\mathbf{w}))+\gamma(\mathbf{v})\alpha(\mathbf{u})\beta(\mathbf{w}))\\
&\quad -\gamma(\mathbf{v})\beta(\mathbf{u})\alpha(\mathbf{w}))-\beta(\mathbf{v})\alpha(\mathbf{u})\gamma(\mathbf{w}))-\alpha(\mathbf{v})\gamma(\mathbf{u})\beta(\mathbf{w}))]\\
&= \alpha(\mathbf{v})\frac{1}{2!}[\beta(\mathbf{u})\gamma(\mathbf{w})-\beta(\mathbf{w})\gamma(\mathbf{u})]\\
&\quad -\alpha(\mathbf{u})\frac{1}{2!}[\beta(\mathbf{v})\gamma(\mathbf{w})-\beta(\mathbf{w})\gamma(\mathbf{v})]\\
&\quad +\alpha(\mathbf{w})\frac{1}{2!}[\beta(\mathbf{v})\gamma(\mathbf{u})-\beta(\mathbf{u})\gamma(\mathbf{v})]\\
&= \alpha(\mathbf{v})(\mathbf{u}\wedge\mathbf{w})(\beta,\gamma)-\alpha(\mathbf{u})(\mathbf{v}\wedge\mathbf{w})(\beta,\gamma)\\
&\quad +\alpha(\mathbf{w})(\mathbf{v}\wedge\mathbf{u})(\beta,\gamma).
\end{aligned}
\tag{C.47}
$$

Since β and γ are arbitrary, we conclude the following expression

$$
\begin{aligned}
\alpha\rfloor(\mathbf{v}\wedge\mathbf{u}\wedge\mathbf{w}) &= (\alpha\rfloor\mathbf{v})(\mathbf{u}\wedge\mathbf{w})-(\alpha\rfloor\mathbf{u})(\mathbf{v}\wedge\mathbf{w})+(\alpha\rfloor\mathbf{w})(\mathbf{v}\wedge\mathbf{u})\\
&= \alpha(\mathbf{v})(\mathbf{u}\wedge\mathbf{w})-\alpha(\mathbf{u})(\mathbf{v}\wedge\mathbf{w})+\alpha(\mathbf{w})(\mathbf{v}\wedge\mathbf{u}).
\end{aligned}
\tag{C.48}
$$

Now using Eq. (C.46) for the 2-vector case and the last one for the 3-vector case, together with the associativity of the exterior product, we have that

$$
\alpha\rfloor((\mathbf{v}\wedge\mathbf{u})\wedge\mathbf{w}) = (\alpha\rfloor(\mathbf{v}\wedge\mathbf{u}))\wedge\mathbf{w}+\mathbf{v}\wedge\mathbf{u}(\alpha\rfloor\mathbf{w}), \tag{C.49}
$$

$$
\alpha\rfloor(\mathbf{v}\wedge(\mathbf{u}\wedge\mathbf{w})) = (\alpha\rfloor\mathbf{v})\mathbf{u}\wedge\mathbf{w}-\mathbf{v}\wedge(\alpha\rfloor(\mathbf{u}\wedge\mathbf{w})). \tag{C.50}
$$

Both equations (C.49) and (C.50) considers the contraction of the exterior product between a 1-vector and a 2-vector. One important point to note here is the fact that the presence of either a positive or a negative sign according to respectively to the presence of a 2-vector in Eq. (C.49) and a 1-vector in Eq. (C.50). For the general case, since the grade involution takes into account the different signs, for $A, B \in \bigwedge(V)$ it follows that:

$$\alpha \rfloor (A \wedge B) = (\alpha \rfloor A) \wedge B + \widehat{A} \wedge (\alpha \rfloor B). \tag{C.51}$$

The right contraction is defined analogously as the left contraction. For an arbitrary multivector A, the left and the right contraction are related by [7]

$$\alpha \rfloor A = -\widehat{A} \lfloor \alpha. \tag{C.52}$$