Trabalho de Conclusão de Curso - Bacharelado em Matemática

Clifford Algebras, Spinors and Triality

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Resumo

O objetivo central deste trabalho é primeiramente introduzir os fundamentos relacionados às álgebras de Clifford, suas construções e exemplos clássicos, com o objetivo de identificar os chamados grupos Spin e investigar algumas propriedades subsequentes dos espaços que carregam as representações desses grupos. Desenvolvemos também o estudo da classificação das álgebras de Clifford e suas representações, juntamente com o teorema da periodicidade de Atiyah-Bott-Shapiro. Finalmente, implementamos o princípio da trialidade, que garante a existência de um automorfismo de ordem três que permuta ciclicamente as representações fundamentais (não equivalentes) e a representação adjunta do grupo Spin(8).

Palavras Chaves: Álgebras de Clifford, Espinores, Trialidade

Abstract

This work has as main objective the introduction of the principal aspects related to the Clifford algebras, their construction and classical examples, aiming to identify the so called Spin groups and to investigate some subsequent properties of the spaces that carry these groups representations. Also, we develop the study of Clifford algebras classification and their representations, as well as the Atiyah-Bott-Shapiro Periodicity Theorem. Finally, the triality principle is implemented, which guarantees the existence of an order three automorphism that cyclically permutates the non-equivalent fundamental representations and the adjoint representations of the Spin(8) group.

Keywords: Clifford algebras, Spinors, Triality

Introduction

In 1878, William K. Clifford proposed a novel formulation to build up vector algebras [1] by introducing a new product in the Grassmann (exterior) algebra. Such spaces, which are now called Clifford algebras, encompass the exterior and inner product spaces formalisms, being, therefore, extremely useful in various applications in mathematics, physics and engineering. An important treatment of such algebras was presented by Lipschitz [2], when he introduced the notion of the Spin group to describe the reduced group representations encoded by rotations in $\mathbb{R}^n$. Further on, spinors have prominently emerged in physics through the Pauli theory of non-relativistic quantum mechanics and on Dirac's approach to relativistic wave mechanics as well. In this work, we investigate the construction of the Clifford algebras and the results on two different definitions of spinors: the algebraic and the classical ones, the former being associated with the search for a minimal regular representation space, and the latter to an irreducible representation for the Spin group. We study their classification and the relationship with the so-called Dirac spinors, following up to the triality principle, which we discuss in conclusion.

In the first chapter, we present some results and rudiments on Clifford algebras. These include the direct definition of such spaces and also an explicit construction with respect to the quotient of the tensor algebra by a two-sided ideal. The chapter is concluded with a presentation of the Atiyah-Bott-Shapiro Periodicity Theorem, which allows the classification of every Clifford algebras constructed for the quadratic spaces $\mathbb{R}^{p,q}$.

The second chapter is devoted to spinors, in which we begin by discussing orthogonal transformations and related representations in terms of Clifford algebras. The classification of the algebraic and classical spinors, as well as their relations, are focused, leading to the study of Dirac spinors and some physical realizations. Spinor classification by their bilinear covariants is, at last, implemented.

Finally, the third chapter encompasses the construction of spinor inner products and a formulation of the triality principle in terms of Clifford algebras, where an explicit construction for the triality automorphism is given.

1 Clifford Algebras

In this chapter we present the concepts of algebras over vector fields and study the Clifford algebras constructions and their classification theorems. We remark that several structures seen in linear algebra, such as tensor and exterior algebras, shall be adopted henceforth. A more detailed look at these results can be found in Appendix A, as well as several symbols that shall be used throughout the text. This discussion was precluded so that we could focus on the subject of matter, which we begin to exam in the subsequent section. The following results were extracted from the refs. [3, 4, 5].

1.1 The Quadratic Space Approach

Let V be a vector space equipped with a bilinear form $g : V \times V \to \mathbb{R}$. The pair (V, g) is called a quadratic space if g is a quadratic form, that is, a symmetric bilinear form.

Definition 1.1. *Let $\mathcal{A}$ be a vector space over a field $\mathbb{K}$ and $* : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$, a closed product in $\mathcal{A}$. Then, $\mathcal{A}$ is said to be an algebra if the product $*$ is bilinear, that is:*

1. $\mathbf{u} * (\mathbf{v} + \mathbf{w}) = \mathbf{u} * \mathbf{v} + \mathbf{u} * \mathbf{w}$,
2. $(\mathbf{u} + \mathbf{v}) * \mathbf{w} = \mathbf{u} * \mathbf{w} + \mathbf{v} * \mathbf{w}$,
3. $a(\mathbf{u} * \mathbf{v}) = (a\mathbf{u}) * \mathbf{v} = \mathbf{u} * (a\mathbf{v});\ \ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V,\ \ \forall a \in \mathbb{K}$.

If there is an element $e \in \mathcal{A}$ such that $\mathbf{u} * e = e * \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathcal{A}$, then $\mathcal{A}$ is called an algebra with unity (or unital). Also, it is said to be commutative if $\mathbf{u} * \mathbf{v} = \mathbf{v} * \mathbf{u}$ and associative if $(\mathbf{u} * \mathbf{v}) * \mathbf{w} = \mathbf{u} * (\mathbf{v} * \mathbf{w})$, for every $\mathbf{u}, \mathbf{v} \in \mathcal{A}$.

Definition 1.2. *The pair $(\mathcal{A}, \gamma)$ is a Clifford Algebra (CA) for the quadratic space (V, g) if $\mathcal{A}$ is generated as an unital associative algebra by $\{\gamma(\mathbf{v}) : \mathbf{v} \in V\}$ and $\{a1_{\mathcal{A}} : a \in \mathbb{R}\}$, where $1_{\mathcal{A}}$ is its unity. Also, γ must satisfy*

$$\gamma(\mathbf{v})\gamma(\mathbf{u}) + \gamma(\mathbf{u})\gamma(\mathbf{v}) = 2g(\mathbf{v}, \mathbf{u})1_{\mathcal{A}}, \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

As it is possible to see, for $\mathbf{v} \in V$ there holds

$$\gamma(\mathbf{v})^2 = Q(\mathbf{v}) = g(\mathbf{v},\mathbf{v}), \tag{1.1.1}$$

and therefore γ acts as a kind of square root of the quadratic form $Q(\mathbf{v}) = g(\mathbf{v},\mathbf{v})$. The application γ is called a *Clifford application*. Consider now an orthonormal basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ for V and $(\mathcal{A},\gamma)$ a Clifford algebra for (V,g). Then,

$$\gamma(\mathbf{e}_i)\gamma(\mathbf{e}_j) + \gamma(\mathbf{e}_j)\gamma(\mathbf{e}_i) = 0, \quad (i \neq j) \tag{1.1.2}$$

and

$$(\gamma(\mathbf{e}_i))^2 = g(\mathbf{e}_i,\mathbf{e}_i)1_{\mathcal{A}}. \tag{1.1.3}$$

Since $\mathcal{A}$ is generated as an algebra by $\{\gamma(\mathbf{v}) : \mathbf{v} \in V\}$ and $\{a1_{\mathcal{A}} : a \in \mathbb{R}\}$, then it is spanned by the products

$$\mathcal{A} = \text{span}\,\{\gamma(\mathbf{e}_1)^{\mu_1}\gamma(\mathbf{e}_2)^{\mu_2}\cdots\gamma(\mathbf{e}_n)^{\mu_n} : \mu_i = 0,1\}, \tag{1.1.4}$$

where we denote $\gamma(\mathbf{e}_1)^0\ldots\gamma(\mathbf{e}_n)^0 = 1_{\mathcal{A}}$. Notice that there are at most 2^n elements written as

$$\gamma(\mathbf{e}_1)^{\mu_1}\cdots\gamma(\mathbf{e}_n)^{\mu_n}, \quad (\mu_i = 0,1) \tag{1.1.5}$$

and therefore, the maximal dimension of a Clifford algebra is 2^n.

Definition 1.3. *Let $(\mathcal{A},*)$ and $(\mathcal{B},\cdot)$ be algebras. An algebra homomorphism is a linear transformation $\phi : \mathcal{A} \to \mathcal{B}$ such that for every $a,b \in \mathcal{A}$ there holds $\phi(a*b) = \phi(a)\cdot\phi(b)$.*

Definition 1.4. *A Clifford algebra $(\mathcal{A},\gamma)$ for the quadratic space (V,g) is called an universal Clifford algebra if for each CA $(\mathcal{B},\rho)$ for (V,g) there is an algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ such that $\rho = \phi\circ\gamma$ and $\phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. The universal Clifford algebra for (V,g) is denoted by $\mathcal{C}\ell(V,g)$.*

Proposition 1.5. *Let $(\mathcal{A},\gamma)$ be a CA for the quadratic space (V,g). If $\dim\mathcal{A} = 2^{\dim V}$, then $(\mathcal{A},\gamma)$ is an universal Clifford algebra for the quadratic space (V,g).*

Proof. Let $B = \{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ be an orthonormal basis for V. For each Clifford algebra $(\mathcal{A},\gamma)$, one has $\gamma(\mathbf{e}_i)\gamma(\mathbf{e}_j) + \gamma(\mathbf{e}_j)\gamma(\mathbf{e}_i) = 0$, for $(i \neq j)$ and $(\gamma(\mathbf{e}_i))^2 = g(\mathbf{e}_i,\mathbf{e}_i)1_{\mathcal{A}}$. Our hypothesis is that $\dim\mathcal{A} = 2^n$. That way, not only does $\{\gamma(\mathbf{e}_1)^{\mu_1}\cdots\gamma(\mathbf{e}_n)^{\mu_n} : \mu_i = 0,1\}$ span $\mathcal{A}$, but it is also a basis for $\mathcal{A}$. Let now $(\mathcal{B},\rho)$ be an arbitrary CA. It follows that $\rho(\mathbf{e}_i)\rho(\mathbf{e}_j) + \rho(\mathbf{e}_j)\rho(\mathbf{e}_i) = 0$, $(i \neq j)$ and $(\rho(\mathbf{e}_i))^2 = g(\mathbf{e}_i,\mathbf{e}_i)1_{\mathcal{A}}$, where the set

$\{\rho(\mathbf{e}_1)^{\mu_1}\rho(\mathbf{e}_2)^{\mu_2}\cdots\rho(\mathbf{e}_n)^{\mu_n} : \mu_i = 0,1\}$ generates $\mathcal{B}$. Now, define $\phi : \mathcal{A} \to \mathcal{B}$ by

$$\phi(\gamma(\mathbf{e}_1)^{\mu_1}\gamma(\mathbf{e}_2)^{\mu_2}\cdots\gamma(\mathbf{e}_n)^{\mu_n}) = \rho(\mathbf{e}_1)^{\mu_1}\rho(\mathbf{e}_2)^{\mu_2}\cdots\rho(\mathbf{e}_n)^{\mu_n}.$$

That way defined, ϕ is an linear application satisfying

$$\phi(\gamma(\mathbf{e}_i))\phi(\gamma(\mathbf{e}_j)) + \phi(\gamma(\mathbf{e}_j))\phi(\gamma(\mathbf{e}_i)) = 2g(\mathbf{e}_i,\mathbf{e}_j)1_{\mathcal{B}},$$

which concludes the proof. □

1.2 The Two-Sided Ideal Approach

Let (V,g) be a quadratic space and $T(V)$ the underlying contravariant tensor algebra. Consider the ideal I of $T(V)$ generated by elements in the form $\mathbf{v}\otimes\mathbf{v} - Q(\mathbf{v})1$, that is:

$$I = \langle \mathbf{v}\otimes\mathbf{v} - Q(\mathbf{v})1,\ \ \forall\mathbf{v}\in V\rangle, \tag{1.2.1}$$

where $Q(\mathbf{v}) = g(\mathbf{v},\mathbf{v})$ and 1 is the tensor algebra identity. Then, one can prove the following statement:

Theorem 1.6. *Let (V,g) be a quadratic space and let I be the two-sided ideal $I = \langle\mathbf{v}\otimes\mathbf{v} - Q(\mathbf{v})1,\ \ \forall\mathbf{v}\in V\rangle$. Then, the quotient space $T(V)/I$ is a Clifford algebra for (V,g). Moreover, if $[\mathbf{v}],[A_{[p]}]\in T(V)/I$, where $\mathbf{v}\in V$ and $A_{[p]}\in\Lambda_p(V)$, then the Clifford algebra product is given by*

$$\mathbf{v}A_{[p]} = \mathbf{v}\wedge A_{[p]} + \mathbf{v}_\flat\lrcorner A_{[p]}, \tag{1.2.2}$$

where $\wedge$ and $\lrcorner$ are the exterior product and the left contraction, respectively[1]

The proof is constructive and rather long, so it can be found in Appendix B. We remark that in Eq.(1.2.2) we have indeed omitted the bracket notation, since it is not necessary, as it can be seen throughout the proof. For examples, see ref. [3], where explicit constructions are given to fully appreciate the underlying geometric meaning of the Clifford product.

We shall henceforth only focus on the Clifford algebras[2] $\mathcal{C}\ell(V,g)$, considering the consequences of the above construction, taking into account the tensor algebra quotient formulation and the product given by Theorem 1.6.

[1] In Eq.(1.2.2), the *musical isomorphism* $(\,\cdot\,)_\flat : V\to V^*$, where $\mathbf{v}_\flat(\mathbf{u}) = g(\mathbf{v},\mathbf{u})$ is used. More details can be found in ref. [6].

[2] The word *universal* is suppressed hereinafter for simplicity.

Consider an orthogonal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for V. Then, since $g(\mathbf{e}_i, \mathbf{e}_j) = 0$ for every $i \neq j$, it follows from Eq.(1.2.2) that

$$\mathbf{e}_i \mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j. \tag{1.2.3}$$

More generally, this equation can be used as many time as we want to produce the relation

$$\mathbf{e}_{\mu_1} \cdots \mathbf{e}_{\mu_p} = \mathbf{e}_{\mu_1} \wedge \cdots \wedge \mathbf{e}_{\mu_p}, \quad (\mu_1 \neq \cdots \neq \mu_p). \tag{1.2.4}$$

Consequently, there is a natural way to associate the Clifford and exterior algebras generators. This yields a (vector space) isomorphism between $T(V)/I$ and $\Lambda(V)$, that is

$$T(V)/I \simeq \Lambda(V). \tag{1.2.5}$$

Then, the dimension of $T(V)/I$ is given by

$$\dim T(V)/I = 2^{\dim V}, \tag{1.2.6}$$

and it follows from Proposition 1.5 that $T(V)/I = \mathcal{C}\ell(V,g)$. Moreover, one can explicitly prove:

Theorem 1.7. *Let (V,g) be a quadratic space, $T(V)/I$ the Clifford algebra as constructed above and $(\mathcal{A},\rho)$ any Clifford algebra for the space (V,g). Then, there is a linear application $\phi : T(V)/I \to \mathcal{A}$ such that $\rho = \phi \circ \gamma$, where γ is the Clifford application $\gamma : V \to T(V)/I$.*

Proof. First, recall that $(\rho(\mathbf{v}))^2 = Q(\mathbf{v})$, where $Q(\mathbf{v}) = g(\mathbf{v},\mathbf{v})$. Now, one can extend ρ to $T(V)$ by defining a linear application $\rho' : T(V) \to \mathcal{A}$ such that

$$\rho'(\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_k) = \rho(\mathbf{v}_1) \cdots \rho(\mathbf{v}_k), \tag{1.2.7}$$

which is possible due to the fact that $\mathcal{L}(T_k(V);\mathcal{A}) \simeq \mathcal{L}(V,\ldots,V;\mathcal{A})$, where for each finite tensor product of elements $\mathbf{v}_1,\ldots,\mathbf{v}_n \in V$, the map ρ' sends $\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n$ to the image through the isomorphism $(\mathbf{v}_1,\ldots,\mathbf{v}_n) \mapsto \rho(\mathbf{v}_1)\ldots\rho(\mathbf{v}_n)$. Consider now the quotient space $T(V)/\ker\,\rho'$. Let $\pi : T(V) \to T(V)/\ker\,\rho'$ be the natural quotient mapping $\pi(x) = [x]$, where $x \in T(V)$. We know then that there is a homomorphism $\phi : T(V)/\ker\,\rho' \to \mathcal{A}$ such that

$$\phi([x]) = \rho'(x). \tag{1.2.8}$$

Furthermore, notice that

$$\rho'(\mathbf{v}\otimes\mathbf{v}-Q(\mathbf{v})1)=\rho'(\mathbf{v}\otimes\mathbf{v})-\rho'(Q(\mathbf{v})1)=(\rho(\mathbf{v}))^2-Q(\mathbf{v})=0, \tag{1.2.9}$$

and therefore $I\subset Ker\ \rho'$. It follows that $T(V)/I\subset T(V)/\ker\ \rho'$. We can then restrict the homomorphism $\phi : T(V)/I \to \mathcal{A}$ and it follows that for $\mathbf{v}\in V$, $\phi([\mathbf{v}])=\rho(\mathbf{v})$, where $[\mathbf{v}]=\gamma(\mathbf{v})$. Therefore,

$$\phi\circ\gamma=\rho, \tag{1.2.10}$$

which concludes the proof that $T(V)/I=\mathcal{C}\ell(V,g)$ is the universal Clifford algebra.

□

It follows that the Clifford algebra inherits the multivector structure from the exterior algebra. We can then write

$$\mathcal{C}\ell(V,g)\simeq\bigoplus_{k=0}^{n}\Lambda_k(V), \tag{1.2.11}$$

where an element $A_{[k]}\in\Lambda_k(V)$ is generally called a k-vector. The anti-automorphisms defined for the exterior algebra[3] can be inherited by the Clifford algebra: the grade involution ($\#$ or $\widehat{(\cdot)}$), reversion ($\sim$) and conjugation all preserve the ideal I shown in Theorem 1.6. Therefore, for elements in the Clifford algebra $A_{[k]}\in\Lambda_k(V)$ there holds

$$\#A_{[p]}=\widehat{A}_{[p]}=(-1)^pA_{[p]}, \tag{1.2.12}$$

whereas for the reversion,

$$\widetilde{A_{[p]}B_{[q]}}=\widetilde{B_{[q]}}\widetilde{A_{[p]}}, \tag{1.2.13}$$

where $\widetilde{A}_{[0]}=A_{[0]}$ and $\widetilde{A}_{[1]}=A_{[1]}$. It follows that

$$\widetilde{A}_{[p]}=(-1)^{\frac{p(p-1)}{2}}A_{[p]}. \tag{1.2.14}$$

Finally, the conjugation is given by

$$\overline{A}_{[p]}=\widehat{\widetilde{A_{[p]}}}=\widetilde{\widehat{A_{[p]}}}. \tag{1.2.15}$$

Besides, one can define the *k-projector*:

$$\langle\cdot\rangle_k:\mathcal{C}\ell(V,g)\to\Lambda_k(V), \tag{1.2.16}$$

[3] More details can be found in Appendix A.

which extracts the k-vector part of an element in the Clifford algebra.

The exterior algebra can be divided into two classes of parity[4], and hence the Clifford algebra inherits such property. Thus, one can write

$$\mathcal{C}\ell(V,g) = \mathcal{C}\ell^{+}(V,g) \oplus \mathcal{C}\ell^{-}(V,g), \tag{1.2.17}$$

where $\mathcal{C}\ell^{+}(V,g)$ and $\mathcal{C}\ell^{-}(V,g)$ are called the even and odd parts of $\mathcal{C}\ell(V,g)$, respectively. Also, the following relations can be derived:

$$\begin{aligned} &\mathcal{C}\ell^{+}(V,g)\mathcal{C}\ell^{+}(V,g) \subset \mathcal{C}\ell^{+}(V,g), \quad &\mathcal{C}\ell^{+}(V,g)\mathcal{C}\ell^{-}(V,g) \subset \mathcal{C}\ell^{-}(V,g),\\ &\mathcal{C}\ell^{-}(V,g)\mathcal{C}\ell^{+}(V,g) \subset \mathcal{C}\ell^{-}(V,g), \quad &\mathcal{C}\ell^{-}(V,g)\mathcal{C}\ell^{-}(V,g) \subset \mathcal{C}\ell^{+}(V,g), \end{aligned} \tag{1.2.18}$$

where $\mathcal{C}\ell^{+}(V,g)$ is a subalgebra of $\mathcal{C}\ell(V,g)$, since

$$\mathcal{C}\ell^{+}(V,g)\mathcal{C}\ell^{+}(V,g) \subset \mathcal{C}\ell^{+}(V,g).$$

We shall now investigate the Clifford algebras for the quadratic spaces $V = \mathbb{R}^n$. Let g be a symmetric bilinear form in $\mathbb{R}^n$ of signature (p,q), where p and q are integers such that $p+q=n$ and such that if $B = \{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ is an orthonormal basis and $\mathbf{v} = \sum_i v^i \mathbf{e}_i$, then

$$g(\mathbf{v},\mathbf{v}) = (v^1)^2 + \cdots + (v^p)^2 - (v^{p+1})^2 + \cdots - (v^n)^2. \tag{1.2.19}$$

Such quadratic spaces are denoted $\mathbb{R}^{p,q}$ and their Clifford algebras are denoted $\mathcal{C}\ell_{p,q}$, that is:

$$\mathcal{C}\ell_{p,q} = \mathcal{C}\ell(\mathbb{R}^{p,q}). \tag{1.2.20}$$

The center $Cen(\mathcal{C}\ell_{p,q})$ of the Clifford algebra $\mathcal{C}\ell_{p,q}$ is defined as the set of elements in $\mathcal{C}\ell_{p,q}$ that commute with all other elements in $\mathcal{C}\ell_{p,q}$, that is:

$$Cen(\mathcal{C}\ell_{p,q}) = \{a \in \mathcal{C}\ell_{p,q} : ax = xa,\ \forall x \in \mathcal{C}\ell_{p,q}\}. \tag{1.2.21}$$

We can prove, then, the two following propositions.

Proposition 1.8.

$$Cen(\mathcal{C}\ell_{p,q}) = \begin{cases} \Lambda_0(\mathbb{R}^{p,q}) & \text{if } n \text{ is even,}\\ \Lambda_0(\mathbb{R}^{p,q}) \oplus \Lambda_n(\mathbb{R}^{p,q}) & \text{if } n \text{ is odd.} \end{cases}$$

[4]Such algebras are called $\mathbb{Z}_2$-graded algebras.

Proof. First, regardless of the parity of n, it is clear that $\Lambda_0(\mathbb{R}_{p,q}) \subset Cen(\mathcal{C}\ell_{p,q})$. Furthermore, $\Lambda_n(\mathbb{R}_{p,q}) \subset Cen(\mathcal{C}\ell_{p,q})$ if, and only if, for every $\mathbf{e}_i$ of the basis in V one has

$$\mathbf{e}_i(\mathbf{e}_1 \cdots \mathbf{e}_n) = (\mathbf{e}_1 \cdots \mathbf{e}_n)\mathbf{e}_i. \tag{1.2.22}$$

Also, by the Clifford algebra rule

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \quad (i \neq j). \tag{1.2.23}$$

Then, we can derive

$$\mathbf{e}_i(\mathbf{e}_1 \cdots \mathbf{e}_n) = (-1)^{n-1}(\mathbf{e}_1 \cdots \mathbf{e}_n)\mathbf{e}_i, \tag{1.2.24}$$

and it follows that that Eq.(1.2.22) is satisfied if and only if n is odd.

Now, let $I, J \subset \{1, \dots, n\}$ be sets such that $\mathbf{e}_I = \prod_{i\in I} \mathbf{e}_i \in Cen(\mathcal{C}\ell_{p,q})$ and $\mathbf{e}_J = \prod_{j\in J} \mathbf{e}_j$. We shall find the cardinality of the set I. Notice that

$$\mathbf{e}_I\mathbf{e}_J = (-1)^{|I||J|-|I\cap J|}\mathbf{e}_J\mathbf{e}_I. \tag{1.2.25}$$

This is true because for each $i \in I$, $\mathbf{e}_i$ anti-commutates with $|J|$ elements in $\mathbf{e}_J$. Since this is done $|I|$ times and since commutation (instead of anti-commutation) holds every time $\mathbf{e}_i$ has to switch with the same (possible) basis element in $\mathbf{e}_J$, we have the formula above. Now, if I is empty we can then regard $\mathbf{e}_I$ as the unity in $\mathbb{R}$ and there is nothing to prove. On the other hand, suppose that I is a proper subset of $\{1, \dots, n\}$, let us say $I = \{i_1, \dots, i_k\}$. Since Eq.(1.2.25) is valid for any J, let $J = \{i_1, i_0\}$, where $i_0 \notin I$. Then, it follows that

$$|I||J| - |I \cap J| = 2k - 1,$$

which means that the commutation does not hold. Since this is a contradiction, one must have $I = \{1, \dots, n\}$. Finally, as stated, if n is even then the only possibility is $\mathbf{e}_I \in \Lambda_0(\mathbb{R}_{p,q})$, and if n is odd, it must be that $\mathbf{e}_I \in \Lambda_0(\mathbb{R}_{p,q}) \oplus \Lambda_n(\mathbb{R}_{p,q})$. □

Proposition 1.9. *If $a \in \mathcal{C}\ell_{p,q} \cap \Lambda_k(\mathbb{R}^{p,q})$ is non-null and $a\mathbf{v} = -\mathbf{v}a$ for every $\mathbf{v} \in \mathbb{R}^{p,q}$, then $a \in \mathcal{C}\ell^+_{p,q}$.*

Proof. It suffices to prove that k is even. Suppose, by contradiction, that k is odd. Then, by Eq.(1.2.2) we have

$$a\mathbf{v} = a \wedge \mathbf{v} + a \lfloor (\mathbf{v})_\flat = -\mathbf{v} \wedge a - (\mathbf{v})_\flat \rfloor a = -\mathbf{v}a. \tag{1.2.26}$$

Notice that by Eq.(5.1.15) we have $a \wedge \mathbf{v} = (-1)^k \mathbf{v} \wedge a$, and since k is odd it follows that $a \wedge \mathbf{v} + \mathbf{v} \wedge a = 0$. Then, Eq.(5.1.48) implies that

$$a \lfloor (\mathbf{v})_\flat + (\mathbf{v})_\flat \rfloor a = a \rfloor (\mathbf{v})_\flat (1 + (-1)^{k-1}) = 0. \tag{1.2.27}$$

Since k is odd, we have $(1 + (-1)^{k-1}) \neq 0$, which implies that

$$a \lfloor (\mathbf{v})_\flat = 0. \tag{1.2.28}$$

Since the contraction of an element $a \in \Lambda_k(\mathbb{R}^{p,q})$ by a vector $\mathbf{v}$ has its image in the $\Lambda_{k-1}(\mathbb{R}^{p,q})$, it follows that $k = 1$. But then it would be true that for every $\mathbf{v} \in \mathbb{R}^{p,q}$

$$a \lfloor (\mathbf{v})_\flat = g(a, \mathbf{v}) = 0. \tag{1.2.29}$$

Since the quadratic space $\mathbb{R}^{p,q}$ is non-degenerate and $a \neq 0$, we arrive at a contradiction. Therefore, k must be even. □

1.3 Classification and Representation

Let V be a real vector space. The *complexification* $V_\mathbb{C}$ of V is the space of elements in the form $\mathbf{v} + i\mathbf{u}$, where $\mathbf{v}, \mathbf{u} \in V$ and i is the imaginary unit. One defines the vector space sum and the product by a complex scalar $(a + bi)$ as it follows:

$$(\mathbf{v}_1 + i\mathbf{u}_1) + (\mathbf{v}_2 + i\mathbf{u}_2) = (\mathbf{v}_1 + \mathbf{v}_2) + i(\mathbf{u}_1 + \mathbf{u}_2),$$

$$(a + ib)(\mathbf{v} + i\mathbf{u}) = (a\mathbf{v} - b\mathbf{u}) + i(b\mathbf{v} + a\mathbf{u}).$$

The dimension of $V_\mathbb{C}$ over $\mathbb{C}$ is $dim_\mathbb{C}\, V_\mathbb{C} = n$, whereas $dim_\mathbb{R}\, V_\mathbb{C} = 2n$. It can be seen that

$$V_\mathbb{C} = \mathbb{C} \otimes V.$$

Additionally, given g a symmetric bilinear form in V, the complex extension $g_\mathbb{C}$ for g can be defined as

$$g_\mathbb{C}(\mathbf{v}_1 + i\mathbf{u}_1, \mathbf{v}_2 + i\mathbf{u}_2) = g(\mathbf{v}_1, \mathbf{v}_2) - g(\mathbf{u}_1, \mathbf{u}_2) + i(g(\mathbf{v}_1, \mathbf{u}_2) + g(\mathbf{u}_1, \mathbf{v}_2)).$$

Theorem 1.10. *Let (V, g) be a quadratic space over $\mathbb{R}$ and $\mathcal{C}\ell(V, g)$ its real Clifford algebra. Consider $\mathcal{C}\ell(V_\mathbb{C}, g_\mathbb{C})$, the complex Clifford algebra for the quadratic complexified* *space* $(V_{\mathbb{C}}, g_{\mathbb{C}})$*. Then*

$$\mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) \simeq \mathcal{C}\ell_{\mathbb{C}}(V, g), \tag{1.3.1}$$

where $\mathcal{C}\ell_{\mathbb{C}}(V,g) = \mathbb{C} \otimes \mathcal{C}\ell(V,g)$ *is the complexification of* $\mathcal{C}\ell(V,g)$*.*

This theorem is shown in Appendix C. This results shows that to describe the structure of Clifford algebras for the quadratic spaces $\mathbb{R}^{p,q}$ and their complexifications, it suffices that we study the structure of the *real* Clifford algebras. By having the structure of a real CA, the structure of the complex CA can be obtained through complexification. With that in mind, we present the following important isomorphisms for $\mathcal{C}\ell_{p,q}$, as seen in [5].

Theorem 1.11. *Let* $\mathcal{C}\ell_{p,q}$ *be the Clifford algebra for the quadratic space* $\mathbb{R}^{p,q}$*. Then,*

$$\begin{aligned} \mathcal{C}\ell_{p+1,q+1} &\simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{p,q} \\ \mathcal{C}\ell_{q+2,p} &\simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{p,q} \\ \mathcal{C}\ell_{q,p+2} &\simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{p,q} \end{aligned} \tag{1.3.2}$$

where $p > 0$ *or* $q > 0$ *and* $\otimes$ *is the usual tensor product.*

Proof. Let U be a bidimensional space where the symmetric bilinear form g_U is defined in a way that, for an orthonormal base $\{\mathbf{f}_1, \mathbf{f}_2\}$ and $\mathbf{u} = u^1\mathbf{f}_1 + u^2\mathbf{f}_2 \in U$, it follows that

$$g_U = \lambda_1 (u^1)^2 + \lambda_2 (u^2)^2, \;\; \lambda_1 = \pm 1, \; \lambda_2 = \pm 1, \tag{1.3.3}$$

where we choose the values λ_1 and λ_2 accordingly for each case $U = \mathbb{R}^{2,0}$, $\mathbb{R}^{1,1}$ or $\mathbb{R}^{0,2}$. Let $\rho : U - \to \mathcal{C}\ell(U, g_U)$ and $\gamma : \mathbb{R}^{p,q} \to \mathcal{C}\ell_{p,q}$ be Clifford applications. That way, one can define the linear application $\Gamma : \mathbb{R}^{p,q} \otimes U \to \mathcal{C}\ell(U, g_U) \otimes \mathcal{C}\ell_{p,q}$ by

$$\Gamma(\mathbf{v} + \mathbf{u}) = \mathbf{f}_1\mathbf{f}_2 \otimes \mathbf{v} + \mathbf{u} \otimes 1, \;\; \forall \mathbf{u} \in U, \; \mathbf{v} \in \mathbb{R}^{p,q}, \tag{1.3.4}$$

where, for the sake of notation, we made use of the relations $\mathbf{f}_1 = \rho(\mathbf{f}_1)$, $\mathbf{f}_2 = \rho(\mathbf{f}_2)$, $\mathbf{u} = \rho(\mathbf{u})$ and $\mathbf{v} = \gamma(\mathbf{v})$. Then, Γ is also a Clifford application. Indeed:

$$\begin{aligned} (\Gamma(\mathbf{v} + \mathbf{u}))^2 &= (\mathbf{f}_1\mathbf{f}_2 \otimes \mathbf{v} + \mathbf{u} \otimes 1)(\mathbf{f}_1\mathbf{f}_2 \otimes \mathbf{v} + \mathbf{u} \otimes 1) \\ &= (\mathbf{f}_1\mathbf{f}_2)^2 \otimes (\mathbf{v})^2 + (\mathbf{u}\mathbf{f}_1\mathbf{f}_2 + \mathbf{f}_1\mathbf{f}_2\mathbf{u}) \otimes \mathbf{v} + (\mathbf{u})^2 \otimes 1. \end{aligned} \tag{1.3.5}$$

Since $\{\mathbf{f}_1, \mathbf{f}_2\}$ is orthonormal, it follows that $\mathbf{f}_1\mathbf{f}_2 = \mathbf{f}_1 \wedge \mathbf{f}_2 = -\mathbf{f}_2\mathbf{f}_1$. This fact and Eq.(5.2.24) then yields

$$\mathbf{u}\mathbf{f}_1\mathbf{f}_2 + \mathbf{f}_1\mathbf{f}_2\mathbf{u} = \mathbf{u}(\mathbf{f}_1 \wedge \mathbf{f}_2) + (\mathbf{f}_1 \wedge \mathbf{f}_2)\mathbf{u} = 2\mathbf{u} \wedge \mathbf{f}_1 \wedge \mathbf{f}_2 = 0, \tag{1.3.6}$$

considering that $\mathbf{u}$ is a linear combination of $\mathbf{f}_1$ and $\mathbf{f}_2$. Using this result and $(\mathbf{f}_1\mathbf{f}_2)^2 = -(\mathbf{f}_1)(\mathbf{f}_2) = -\lambda_1\lambda_2$, it follows that

$$\begin{aligned}(\Gamma(\mathbf{v}+\mathbf{u}))^2 &= -\lambda_1\lambda_2 \otimes g(\mathbf{v},\mathbf{v}) + g_V(\mathbf{u},\mathbf{u})\otimes 1\\ &= \lambda_1(u^1)^2 + \lambda_2(u^2)^2 \\ &\quad -\lambda_1\lambda_2((v^1)^2+\cdots+(v^p)^2-(v^{p+1})^2-\cdots-(v^n)^2)1\otimes 1.\end{aligned} \tag{1.3.7}$$

Therefore, Γ is a Clifford application $\Gamma : \mathbb{R}^{p,q}\oplus U \to \mathcal{C}\ell(W, g_W)$, where W is a $(n+2)$-dimensional space equipped with a symmetric bilinear form given by

$$g_W(\mathbf{w},\mathbf{w}) = \lambda_1(u^1)^2 + \lambda_2(u^2)^2 - \lambda_1\lambda_2((v^1)^2+\cdots+(v^p)^2-(v^{p+1})^2-\cdots-(v^n)^2), \tag{1.3.8}$$

where $w = u^1\mathbf{f}_1 + u^2\mathbf{f}_2 + v^i\mathbf{e}_i$. Hence, if $U = \mathbb{R}^{2,0}$, it follows that $W = \mathbb{R}^{q+2,p}$, whereas $U = \mathbb{R}^{1,1}$ yields $W = \mathbb{R}^{p+1,q+1}$ and if $U = \mathbb{R}^{0,2}$ then $W = \mathbb{R}^{q,p+2}$. The isomorphisms follow from the universality of the Clifford algebra. □

Combining these isomorphisms, one can obtain several other results. Notice that the repeated use of the first relation in Eq.(1.3.2) yields

$$\mathcal{C}\ell_{p,p} \simeq \otimes^p \mathcal{C}\ell_{1,1}. \tag{1.3.9}$$

Similarly, one can deduce

$$\mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell_{p,p}\otimes\mathcal{C}\ell_{0,q-p}\ \ (q>p), \quad \mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell_{q,q}\otimes\mathcal{C}\ell_{p-q,0}\ \ (p>q). \tag{1.3.10}$$

Furthermore, using either of the other two relations in Eq.(1.3.2) it is possible to derive some interesting particular cases:

$$\mathcal{C}\ell_{0,2}\otimes\mathcal{C}\ell_{2,0} \simeq \mathcal{C}\ell_{0,4}, \quad \mathcal{C}\ell_{0,4}\otimes\mathcal{C}\ell_{4,0} \simeq \mathcal{C}\ell_{0,8} \tag{1.3.11}$$

and

$$\mathcal{C}\ell_{2,2} \simeq \mathcal{C}\ell_{0,2}\otimes\mathcal{C}\ell_{0,2} \simeq \mathcal{C}\ell_{1,1}\otimes\mathcal{C}\ell_{1,1}, \tag{1.3.12}$$

results which imply that

$$\mathcal{C}\ell_{0,4}\otimes\mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell_{p,q+4}, \quad \mathcal{C}\ell_{0,8}\otimes\mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell_{p,q+8}. \tag{1.3.13}$$

Another isomorphism which is not a result of the above is

Lemma 1.12. $\mathcal{C}\ell_{2,0} \simeq \mathcal{C}\ell_{1,1}$.

Proof. Elements in $\mathcal{C}\ell_{2,0}$ are written in the form

$$\mathcal{C}\ell_{2,0} \ni a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_{12}\mathbf{e}_1\mathbf{e}_2,$$

where $(\mathbf{e}_1)^2 = 1$ and $(\mathbf{e}_2)^2 = 1$. Now, for $\mathcal{C}\ell_{1,1}$ we have:

$$\mathcal{C}\ell_{1,1} \ni b_0 + b_1\mathbf{f}_1 + b_2\mathbf{f}_2 + b_{12}\mathbf{f}_1\mathbf{f}_2,$$

where $(\mathbf{f}_1)^2 = 1$ and $(\mathbf{f}_2)^2 = -1$. Then, the linear application $\varphi : \mathcal{C}\ell_{2,0} \to \mathcal{C}\ell_{1,1}$ can be defined as

$$\varphi(1) = 1,\ \varphi(\mathbf{e}_1) = \mathbf{f}_1,\ \varphi(\mathbf{e}_2) = \mathbf{f}_1\mathbf{f}_2,\ \varphi(\mathbf{e}_1\mathbf{e}_2) = \mathbf{f}_2,$$

which is an algebra isomorphism between the spaces. □

Lemma 1.13. $\mathcal{C}\ell_{p+1,q} \simeq \mathcal{C}\ell_{q+1,p}$.

Proof. By Eq.(1.3.2) and Lemma 1.12 it follows that for every integers p and q' there holds

$$\mathcal{C}\ell_{p-1,q'+1} \simeq \mathcal{C}\ell_{q'+2,p}. \tag{1.3.14}$$

Just set $q = q' - 1$ and we are done. □

Lemma 1.14. $\mathcal{C}\ell^+_{p,q} \simeq \mathcal{C}\ell_{q,p-1} \simeq \mathcal{C}\ell_{p,q-1} \simeq \mathcal{C}\ell^+_{q,p}$.

Proof. Let $\{\mathbf{e}_i, \mathbf{f}_k\}$ $(i = 1,\ldots,p;\ k = 1,\ldots,q)$ be an orthonormal basis for $\mathbb{R}^{p,q}$. Then, (suppressing the Clifford application notation) $\mathcal{C}\ell_{p,q}$ is generated by 1 and $\{\mathbf{e}_i, \mathbf{f}_k\}$ in such a way that we have $(\mathbf{e}_i)^2 = 1$, $(\mathbf{f}_k)^2 = -1$ and the anti-commutation relations: $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0$ $(i \neq j)$, $\mathbf{f}_k\mathbf{f}_l + \mathbf{f}_k\mathbf{f}_j = 0$ $(k \neq l)$ and $\mathbf{e}_i\mathbf{f}_k + \mathbf{f}_k\mathbf{e}_i = 0$. Clearly, the space $\bigwedge_2(\mathbb{R}^{p,q})$ is generated by the set $\{\mathbf{e}_i\mathbf{e}_j\ (i \neq j);\ \mathbf{f}_k\mathbf{f}_l\ (k \neq l);\ \mathbf{e}_i\mathbf{f}_k\}$. However, this set is not the smallest one with such property: notice that

$$(\mathbf{e}_i\mathbf{f}_k)(\mathbf{e}_i\mathbf{f}_l) = -(\mathbf{e}_i)^2\mathbf{f}_k\mathbf{f}_l = -\mathbf{f}_k\mathbf{f}_l\ (k \neq l), \tag{1.3.15}$$

which means that some elements can be written as a product of other elements in the same set. In order to surpass this, one can choose an element (take $\mathbf{e}_1$, for instance), in favor to see that

$$\{\mathbf{e}_1\mathbf{e}_m, \mathbf{e}_1\mathbf{f}_k :\ m = 2,\ldots p,\ k = 1,\ldots,q\} \tag{1.3.16}$$

generates $\bigwedge_2(\mathbb{R}^{p,q})$ and, therefore, generates the subalgebra $\mathcal{C}\ell^+_{p,q}$. That way, one can write the generators in the following manner:

$$\begin{aligned} \phi_a &= \mathbf{e}_1\mathbf{e}_{a+1}, \quad a = 1,\ldots,p-1, \\ \psi_b &= \mathbf{e}_1\mathbf{f}_b, \quad b = 1,\ldots,q. \end{aligned} \tag{1.3.17}$$

Notice here that $(\phi_a)^2 = -(\mathbf{e}_1)^2(\mathbf{e}_{a+1})^2 = -1$ and $(\psi_b)^2 = -(\mathbf{e}_1)^2(\mathbf{f}_b)^2 = 1$. Furthermore, every anti-commutation relation holds for these elements. Therefore, $\{\phi_a, \psi_b\}$ is the set of generators of and Clifford algebra associated to the quadratic space $\mathbb{R}^{q,p-1}$. It follows that

$$\mathcal{C}\ell^+_{p,q} \simeq \mathcal{C}\ell_{q,p-1}. \tag{1.3.18}$$

Finally, the isomorphism $\mathcal{C}\ell_{p+1,q} \simeq \mathcal{C}\ell_{q+1,p}$ can be used to derive the other relations, as wanted. □

We proceed to find explicit isomorphisms for $\mathcal{C}\ell_{1,0}$, $\mathcal{C}\ell_{0,1}$, $\mathcal{C}\ell_{0,2}$ and $\mathcal{C}\ell_{1,1} \simeq \mathcal{C}\ell_{2,0}$, which will be shown to be central in establishing the isomorphisms for all other Clifford algebras $\mathcal{C}\ell_{p,q}$. These isomorphisms shall be regarded later as faithful representations for the Clifford algebras, notion which shall be defined in the end of the chapter.

Lemma 1.15. $\mathcal{C}\ell_{0,1} \simeq \mathbb{C}$.

Proof. An arbitrary element in $\mathcal{C}\ell_{0,1}$ is writen as

$$\mathcal{C}\ell_{0,1} \ni \psi = a + b\mathbf{e}, \tag{1.3.19}$$

where $\mathbf{e}^2 = -1$. This algebra is isomorphic to the complex algebra $\mathbb{C}$, that is, the set of pairs $(a,b) \in \mathbb{R}^2$ endowed with the product

$$(a,b)(c,d) = (ac - bd, ad + bc).$$

The isomorphism between these algebras is $\rho : \mathcal{C}\ell_{0,1} \to \mathbb{C}$, defined by $\rho(1) = (1,0)$ and $\rho(\mathbf{e}) = (0,1) = i$. Therefore

$$\mathcal{C}\ell_{0,1} \simeq \mathbb{C}. \tag{1.3.20}$$

□

Lemma 1.16. $\mathcal{C}\ell_{1,0} \simeq \mathbb{R} \oplus \mathbb{R}$

Proof. An arbitrary element in $\mathcal{C}\ell_{1,0}$ can be written as

$$\mathcal{C}\ell_{1,0} \ni \psi = a + b\mathbf{e}, \tag{1.3.21}$$

and now $\mathbf{e}^2 = 1$. Consider the set[5] $\mathbb{D}$ of elements $(a,b) \in \mathbb{R}^2$, equipped with the following product:

$$(a,b)(c,d) = (ac,bd). \tag{1.3.22}$$

This algebra is the direct sum of real algebras, namely $\mathbb{R}\oplus\mathbb{R}$. It is isomorphic to 2×2 diagonal matrix algebra, where the isomorphism is given by

$$\varphi(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \tag{1.3.23}$$

and since

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix}, \tag{1.3.24}$$

we have that $\mathbb{D} \simeq \mathbb{R}\oplus\mathbb{R}$. As for $\mathcal{C}\ell_{1,0}$, clearly there is an isomorphism $\rho : \mathcal{C}\ell_{1,0} \to \mathbb{D}$ such that $\rho(1) = (1,0)$ and $\rho(\mathbf{e}) = (0,1)$. Therefore, we have

$$\mathcal{C}\ell_{1,0} \simeq \mathbb{R}\oplus\mathbb{R}. \tag{1.3.25}$$

□

Lemma 1.17. $\mathcal{C}\ell_{0,2} \simeq \mathbb{H}$

Proof. Let $\{\mathbf{e}_1,\mathbf{e}_2\}$ be an orthonormal basis for $\mathbb{R}^{0,2}$ such that

$$g(\mathbf{e}_1,\mathbf{e}_1) = g(\mathbf{e}_2,\mathbf{e}_2) = -1,\ g(\mathbf{e}_1,\mathbf{e}_2) = g(\mathbf{e}_2,\mathbf{e}_1) = 0. \tag{1.3.26}$$

An arbitrary element in $\mathcal{C}\ell_{0,2}$ is in the form

$$\mathcal{C}\ell_{0,2} \ni \psi = a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_1\mathbf{e}_2, \tag{1.3.27}$$

with $a,b,c,d \in \mathbb{R}$ and

$$(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = -1,\ \ \mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_2\mathbf{e}_1 = 0. \tag{1.3.28}$$

It follows that

$$(\mathbf{e}_1\mathbf{e}_2)^2 = -1. \tag{1.3.29}$$

Therefore, one can see that $\mathcal{C}\ell_{0,2}$ is isomorphic to the quaternion algebra $\mathbb{H}$. The

[5]The perplex numbers.

isomorphism ρ can be defined, for instance, by

$$\rho(1) = 1,\ \rho(\mathbf{e}_1) = i,\ \rho(\mathbf{e}_2) = j,\ \rho(\mathbf{e}_1\mathbf{e}_2) = k, \tag{1.3.30}$$

where i, j and k are the imaginary units, such that $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Therefore,

$$\mathcal{C}\ell_{0,2} \simeq \mathbb{H}. \tag{1.3.31}$$

□

Lemma 1.18. $\mathcal{C}\ell_{2,0} \simeq \mathcal{C}\ell_{1,1} \simeq \mathcal{M}(2,\mathbb{R})$

Proof. Since the Clifford algebras for the quadratic spaces $\mathbb{R}^{2,0}$ and $\mathbb{R}^{1,1}$ are isomorphic by Lemma 1.12, it suffices to consider only one of them, for instance $\mathcal{C}\ell_{2,0}$. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be an orthonormal base for $\mathbb{R}^{2,0}$ such that

$$g(\mathbf{e}_1,\mathbf{e}_1) = g(\mathbf{e}_2,\mathbf{e}_2) = 1,\ \ g(\mathbf{e}_1,\mathbf{e}_2) = g(\mathbf{e}_2,\mathbf{e}_1) = 0. \tag{1.3.32}$$

An arbitrary element in $\mathcal{C}\ell_{2,0}$ is written as

$$\mathcal{C}\ell_{2,0} \ni \psi = a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_1\mathbf{e}_2, \tag{1.3.33}$$

where $a, b, c, d \in \mathbb{R}$ and

$$(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = 1,\ \ \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0. \tag{1.3.34}$$

We also know that

$$(\mathbf{e}_1\mathbf{e}_2)^2 = -1. \tag{1.3.35}$$

Now, let $\mathcal{M}(2,\mathbb{R})$ be the algebra of real 2×2 matrices. The set

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\} \tag{1.3.36}$$

clearly spans $\mathcal{M}(2,\mathbb{R})$. We can also notice that

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}^2 = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \tag{1.3.37}$$

and therefore we can define ρ as the linear application

$$\begin{aligned} \rho(1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \rho(\mathbf{e}_2) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \rho(\mathbf{e}_1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \rho(\mathbf{e}_1\mathbf{e}_2) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned} \tag{1.3.38}$$

which is clearly an algebra isomorphism. Accordingly, we get that

$$\mathcal{C}\ell_{2,0} \simeq \mathcal{C}\ell_{1,1} \simeq \mathcal{M}(2,\mathbb{R}). \tag{1.3.39}$$

□

Theorem 1.19 (Atiyah-Bott-Shapiro Periodicity Theorem)**.** *For every quadratic space $\mathbb{R}^{p,q}$ it follows that $\mathcal{C}\ell_{p,q+8} \simeq \mathcal{M}(16,\mathbb{R}) \otimes \mathcal{C}\ell_{p,q}$.*

Proof. Using Eq.(1.3.9), Lemma 1.18 and the isomorphism

$$\mathcal{M}(m,\mathbb{R}) \otimes \mathcal{M}(n,\mathbb{R}) \simeq \mathcal{M}(mn,\mathbb{R}), \tag{1.3.40}$$

it follows that

$$\mathcal{C}\ell_{p,p} \simeq \mathcal{M}(2p,\mathbb{R}). \tag{1.3.41}$$

Furthermore, this relation combined with Lemmas 1.17, 1.18 and Eq.(1.3.12) imply that

$$\mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{0,2} \simeq \mathbb{H} \otimes \mathbb{H} \simeq \mathcal{M}(4,\mathbb{R}). \tag{1.3.42}$$

Moreover, Eq.(1.3.11) and Lemmas 1.17, 1.18 can be used to conclude that

$$\mathcal{C}\ell_{0,4} \simeq \mathbb{H} \otimes \mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(2,\mathbb{H}) \simeq \mathcal{C}\ell_{4,0}, \tag{1.3.43}$$

which implies that

$$\begin{aligned} \mathcal{C}\ell_{0,8} &\simeq \mathcal{M}(2,\mathbb{H}) \otimes \mathcal{M}(2,\mathbb{H}) \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{H} \otimes \mathbb{H} \otimes \mathcal{M}(2,\mathbb{R}) \\ &\simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathcal{M}(4,\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(16,\mathbb{R}), \end{aligned} \tag{1.3.44}$$

and finally, using Eq.(1.3.13), it results in

$$\mathcal{C}\ell_{p,q+8} \simeq \mathcal{M}(16,\mathbb{R}) \otimes \mathcal{C}\ell_{p,q}. \tag{1.3.45}$$

□

This last equation shows that we only need to explicitly establish the classifications for Clifford algebras with dimension up to $\dim V = p + q = 8$, since for higher dimensions one can use the isomorphism $\mathcal{C}\ell_{p,q+8} \simeq \mathcal{M}(16,\mathbb{R}) \otimes \mathcal{C}\ell_{p,q}$. For $\dim V \leq 4$, the following isomorphisms hold:

$$\mathcal{C}\ell_{0,0} \simeq \mathbb{R}$$
$$\mathcal{C}\ell_{0,1} \simeq \mathbb{C}$$
$$\mathcal{C}\ell_{1,0} \simeq \mathbb{R} \otimes \mathbb{R}$$
$$\mathcal{C}\ell_{0,2} \simeq \mathbb{H}$$
$$\mathcal{C}\ell_{2,0} \simeq \mathcal{M}(2,\mathbb{R})$$
$$\mathcal{C}\ell_{1,1} \simeq \mathcal{C}\ell_{2,0} \simeq \mathcal{M}(2,\mathbb{R})$$
$$\mathcal{C}\ell_{0,3} \simeq \mathbb{H} \otimes \mathbb{H}$$
$$\mathcal{C}\ell_{3,0} \simeq \mathcal{M}(2,\mathbb{C})$$
$$\mathcal{C}\ell_{1,2} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{0,1} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{C} \simeq \mathcal{M}(2,\mathbb{C})$$
$$\mathcal{C}\ell_{2,1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{1,0} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{R} \otimes \mathbb{R} \simeq \mathcal{M}(2,\mathbb{R} \otimes \mathbb{R})$$
$$\mathcal{C}\ell_{0,4} \simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{2,0} \simeq \mathbb{H} \otimes \mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(2,\mathbb{H})$$
$$\mathcal{C}\ell_{4,0} \simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{H} \simeq \mathcal{M}(2,\mathbb{H})$$
$$\mathcal{C}\ell_{2,2} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{1,1} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(4,\mathbb{R})$$
$$\mathcal{C}\ell_{3,1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{2,0} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathcal{M}(2,\mathbb{R}) \simeq \mathcal{M}(4,\mathbb{R})$$
$$\mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{0,2} \simeq \mathcal{M}(2,\mathbb{R}) \otimes \mathbb{H} \simeq \mathcal{M}(2,\mathbb{H})$$

Other isomorphisms can be seen in ref. [5]. More generally, one can use the Periodicity Theorem to set the following table:

$p-q \mod 8$	0	1	2	3
$\mathcal{C}\ell_{p,q}$	$\mathcal{M}(2^{[n/2]},\mathbb{R})$	$\mathcal{M}(2^{[n/2]},\mathbb{R}) \oplus \mathcal{M}(2^{[n/2]},\mathbb{R})$	$\mathcal{M}(2^{[n/2]},\mathbb{R})$	$\mathcal{M}(2^{[n/2]},\mathbb{C})$
$p-q \mod 8$	4	5	6	7
$\mathcal{C}\ell_{p,q}$	$\mathcal{M}(2^{[n/2]-1},\mathbb{H})$	$\mathcal{M}(2^{[n/2]-1},\mathbb{H}) \oplus \mathcal{M}(2^{[n/2]-1},\mathbb{H})$	$\mathcal{M}(2^{[n/2]-1},\mathbb{H})$	$\mathcal{M}(2^{[n/2]},\mathbb{H})$

Table 1.1: Clifford Algebra Classification ($n = p + q$ and $[\,\cdot\,]$ is the floor function)

Furthermore, it is straightforward to see that since

$$\mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) \simeq \mathbb{C} \otimes \mathcal{C}\ell(V, g), \tag{1.3.46}$$

then the complex case depends only on the parity of $n = p + q$. Therefore,

$n = 2k$	$\mathcal{C}\ell_{\mathbb{C}}(2k) \simeq \mathcal{M}(2^k, \mathbb{C})$
$n = 2k+1$	$\mathcal{C}\ell_{\mathbb{C}}(2k+1) \simeq \mathcal{M}(2^k, \mathbb{C}) \oplus \mathcal{M}(2^k, \mathbb{C})$

Definition 1.20. *Let $\mathcal{A}$ be an algebra and V a vector space over $\mathbb{K}$. A linear application $\rho : \mathcal{A} \to End_{\mathbb{K}}(V)$ satisfying $\rho(1_{\mathcal{A}}) = 1_V$ and $\rho(ab) = \rho(a)\rho(b)$, for every $a, b \in \mathcal{A}$, is called a $\mathbb{K}$-representation of $\mathcal{A}$. Furthermore, V is said to be the representation space of $\mathcal{A}$ with respect to ρ.*

A representation is said to be *faithful* if it is injective (equivalently, if $\ker \rho = \{0\}$) and two representation $\rho_1 : \mathcal{A} \to End_{\mathbb{K}}(V_1)$ and $\rho_2 : \mathcal{A} \to End_{\mathbb{K}}(V_2)$ are *equivalent* if there is a $\mathbb{K}$-isomorphism $\varphi : V_1 \to V_2$ satisfying

$$\rho_2(a) = \phi \circ \rho_1(a) \circ \phi^{-1}, \ \forall a \in \mathcal{A}.$$

Definition 1.21. *Let $\mathcal{A}$ be an algebra and $\rho : \mathcal{A} \to End_{\mathbb{K}}(V)$ a representation. One says that ρ is an irreducible representation if when $I \subset V$ is such that $\rho(a)I \subset I$ for every $a \in \mathcal{A}$, then $I = \{0\}$ or $I = V$.*

Notice here that by the classification isomorphisms, every Clifford algebra has a faithful representation in the form of $\mathcal{M}(n, \mathbb{K}) \simeq \text{End}(\mathbb{K}^n)$, where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Such property shall be crucial to define the concept of algebraic spinors in the next chapter.

2 Spinors

In this chapter, we begin the rudiments to define spinors by analyzing the orthogonal group $O(p,q)$. Further on, algebraic and classical spinors are defined and classified. Finally, we investigate results on Dirac spinors and exhibit a spinorial classification for the complexified Clifford algebra for the Minkwoski space. This work structures the discussion that will be held for the triality principle in Chapter 3. These results and ommited proofs can be found in refs. [3, 4, 5, 7].

2.1 Orthogonal Transformations

Let g be a symmetric bilinear form over V. A linear transformation $T : V \to V$ is called an isometry if

$$g(T(\mathbf{v}), T(\mathbf{u})) = g(\mathbf{v}, \mathbf{u}), \ \forall \mathbf{v}, \mathbf{u} \in V. \tag{2.1.1}$$

Defining T_i^j by $T(\mathbf{e}_i) = T_i^j \mathbf{e}_j$ and $g_{ij} = g(\mathbf{e}_i, \mathbf{e}_j)$, where $\{\mathbf{e}_i\}$ is a basis for V, one can write

$$T_i^k g_{kl} T_j^l = g_{ij} \ or \ T^T G T = G, \tag{2.1.2}$$

where T is a matrix with entries $\{T_j^i\}$. Since $\det(AB) = \det(A) \cdot \det(B)$ and $\det(A) = \det(A^T)$, it follows from the previous equation that

$$\begin{aligned} \det(T^T G T) &= \det(G) \\ \det(T^T) \cdot \det(G) \cdot \det(T) &= \det(G) \\ \det(T^T) \cdot \det(T) &= 1 \\ (\det(T))^2 &= 1. \end{aligned} \tag{2.1.3}$$

Therefore, the possible results are $\det(T) = \pm 1$. When $\det(T) = 1$, T is called a rotation, whereas when $\det(T) = -1$, T is called a reflection. The orthogonal group, $O(p,q)$, is the group of all isometries over $\mathbb{R}^{p,q}$. The special orthogonal group, $SO(p,q)$, consists of rotations in $\mathbb{R}^{p,q}$, that is, isometries T such that $\det(T) = 1$. It is not hard to show that the orthogonal group is not connected, and this result is presented in Appendix D.

Let now V be a vector space over $\mathbb{R}$. A hyperplane in V is a subspace U of V such that $\dim U = \dim V - 1$. A non-isotropic subspace of V is a subspace U such that $g(\mathbf{u},\mathbf{u}) \neq 0$ for all $\mathbf{u} \in U$. Let U be such a subspace of V. We can write $V = U \oplus U^{\perp}$. Then, for every $\mathbf{v} \in V$ we can write $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$ where $\mathbf{v}_{\|} \in U$ and $\mathbf{v}_{\perp} \in U^{\perp}$. Then, the orthogonal symmetry with respect to U is defined as

$$S_U(\mathbf{v}\| + \mathbf{v}_{\perp}) = -\mathbf{v}_{\|} + \mathbf{v}_{\perp}, \ \forall \mathbf{v}_{\|} \in U, \forall \mathbf{v}_{\perp} \in U^{\perp}.$$

Clearly, the matrix S_U associated to this application is such that

$$S_U = -id_U + id_{U^{\perp}}, \tag{2.1.4}$$

which implies that

$$\det(S_U) = (-1)^{\dim U}. \tag{2.1.5}$$

Theorem 2.1 (Cartan-Dieudonné)**.** *Any orthogonal transformation T in a finite dimensional vector space V can be expressed as a finite product of symmetries on non-isotropic hyperplanes.*

The proof is found in Appendix E. Now, in terms of Clifford algebras, let $\mathbf{v}, \mathbf{u} \in \mathcal{C}\ell_{p,q}$ be vectors. By definition, we know that

$$\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u} = 2g(\mathbf{v},\mathbf{u}), \tag{2.1.6}$$

$$\mathbf{u}^2 = g(\mathbf{u},\mathbf{u}), \tag{2.1.7}$$

which implies that if $\mathbf{u}$ is non-isotropic ($g(\mathbf{u},\mathbf{u}) \neq 0$) then

$$\mathbf{u}^{-1} = \frac{\mathbf{u}}{g(\mathbf{u},\mathbf{u})}. \tag{2.1.8}$$

From Eq.(5.5.2), one can write

$$S_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - (\mathbf{v}\mathbf{u} + \mathbf{u}\mathbf{v})\mathbf{u}^{-1} \tag{2.1.9}$$

$$= -\mathbf{u}\mathbf{v}\mathbf{u}^{-1}, \tag{2.1.10}$$

which can also be written as

$$S_{\mathbf{u}}(\mathbf{v}) = \hat{\mathbf{u}}\mathbf{v}\mathbf{u}^{-1}. \tag{2.1.11}$$

This relation will be key to the next section.

2.1.1 Clifford-Lipschitz Groups

A natural group in $\mathcal{C}\ell_{p,q}$ is $\mathcal{C}\ell_{p,q}^{*} = \{a \in \mathcal{C}\ell_{p,q} : \exists a^{-1} \in \mathcal{C}\ell_{p,q}\}$, the invertible elements with respect to the Clifford product. Consider the set $\Gamma_{p,q} = \{a \in \mathcal{C}\ell_{p,q}^{*} : a\mathbf{v}a^{-1} \in \mathbb{R}^{p,q},\ \forall \mathbf{v} \in \mathbb{R}^{p,q}\}$, called the Clifford-Lipschitz group. It is possible to define the so-called *adjoint representation* $\sigma : \Gamma_{p,q} \to \mathrm{Aut}(\mathcal{C}\ell_{p,q})$, given by

$$\sigma(a)(\mathbf{v}) = a\mathbf{v}a^{-1}. \tag{2.1.12}$$

As we shall see, the image of this mapping depends on the parity of $n = p + q$. Aspiring to remove the condition on the space dimension, one can define the *twisted adjoint representation*:

$$\hat{\sigma}(a)(\mathbf{v}) = \hat{a}\mathbf{v}a^{-1}. \tag{2.1.13}$$

Similarly, one can define the twisted Clifford-Lipschitz group by

$$\hat{\Gamma}_{p,q} = \{a \in \mathcal{C}\ell_{p,q}^{*} : \hat{a}\mathbf{v}a^{-1} \in \mathbb{R}^{p,q}, \forall \mathbf{v} \in \mathbb{R}^{p,q}\}. \tag{2.1.14}$$

We have then, the following result:

Theorem 2.2. *Let $\mathcal{C}\ell_{p,q}$ be the Clifford algebra with respect to the quadratic space $\mathbb{R}^{p,q}$ and let $\sigma : \Gamma_{p,q} \to Aut(\mathcal{C}\ell_{p,q})$ and $\hat{\sigma} : \Gamma_{p,q} \to Aut(\mathcal{C}\ell p,q)$ be defined as above. Also, let $\hat{\Gamma}_{p,q}^{+} = \hat{\Gamma}_{p,q} \cap \mathcal{C}\ell_{p,q}^{+}$. Then,*

$$\begin{aligned} &\sigma(\Gamma_{p,q}) = O(p,q), \ \textit{if } n = p+q \textit{ is even},\\ &\sigma(\Gamma_{p,q}) = SO(p,q), \ \textit{if } n = p+q \textit{ is odd},\\ &\hat{\sigma}(\hat{\Gamma}_{p,q}) = O(p,q),\\ &\hat{\sigma}(\hat{\Gamma}_{p,q}^{+}) = SO(p,q). \end{aligned} \tag{2.1.15}$$

The proof can be found in Appendix F. This result can also be used to characterize the Clifford-Lipschitz (as well as the twisted counterpart) as $\Gamma_{p,q} = \{a \in \mathcal{C}\ell_{p,q} : a = \mathbf{u}_1 \dots \mathbf{u}_k,\ \mathbf{u}_i \in \mathbb{R}^{p,q} \text{ and } g(\mathbf{u}_i, \mathbf{u}_i) \neq 0\}$ by Cartan-Dieudonné Theorem, as is can be seen in the proof. We shall simply use the symbol $\Gamma_{p,q}$ to denote both the twisted and regular adjoint representations. Another important result is the following:

Lemma 2.3. *Let $\mathcal{C}\ell_{p,q}$ be the Clifford algebra for the quadratic space $\mathbb{R}^{p,q}$ and let $\hat{\sigma} : \Gamma_{p,q} \to O(p,q)$ be the twisted adjoint representation. Then,* $\ker(\hat{\sigma}) = \mathbb{R}^{*}$.

Proof. By definition, $\ker(\hat{\sigma}) = \{a \in \Gamma_{p,q} : \hat{\sigma}(a) = 1\}$, where 1 denotes the unit in $\mathrm{O}(p,q)$.

Now, let $a \in \ker(\hat{\sigma})$. That way

$$\begin{aligned}\hat{\sigma}(a)(\mathbf{v}) &= \mathbf{v},\\ \hat{a}\mathbf{v}a^{-1} &= \mathbf{v},\\ \hat{a}\mathbf{v} &= \mathbf{v}a.\end{aligned} \tag{2.1.16}$$

Writing $a = a_+ + a_-$, where a_+ denotes the even part and a_- the odd part of a in $\mathcal{C}\ell_{p,q}$, it follows that

$$a_+\mathbf{v} = \mathbf{v}a_+ \qquad a_-\mathbf{v} = -\mathbf{v}a_-. \tag{2.1.17}$$

The second condition in Eq.(2.1.17) is satisfied if and only if $a_- = 0$, since there is no non-null odd element in $\mathcal{C}\ell_{p,q}$ which anti-commutates with all generators $\{\mathbf{e}_i\}$ of $\mathcal{C}\ell_{p,q}$, by Proposition 1.2. For the first equation, it follows that $a_+ \in Cen(\mathcal{C}\ell_{p,q})$, since a_+ commutes with all generators. Now, by Proposition 1.8, if n is even, then $Cen(\mathcal{C}\ell_{p,q}) = \Lambda_0(\mathbb{R}^{p,q})$ and if n is odd, $Cen(\mathcal{C}\ell_{p,q}) = \Lambda_0(\mathbb{R}^{p,q}) \oplus \Lambda_n(\mathbb{R}^{p,q})$. Then, since a_+ cannot be in $\Lambda_n(\mathbb{R}^{p,q})$ in the odd case (because it is the even part of $b^{-1}a$), it must be that $a_+ \in \Lambda_0(\mathbb{R}_{p,q})$. Therefore, the conditions are satisfied if and only if $a \in \Lambda_0(\mathbb{R}^{p,q}) = \mathbb{R}$. Since a must be invertible, it cannot be the zero element and, therefore, it follows that

$$\ker(\hat{\sigma}) = \mathbb{R}^*. \tag{2.1.18}$$

□

2.1.2 The Pin and Spin Groups

It is possible to define a norm for a multivector $a \in \mathcal{C}\ell_{p,q}$ as

$$N(a) = \langle \widetilde{a}a \rangle_0. \tag{2.1.19}$$

Another way of expressing a norm in $\mathcal{C}\ell_{p,q}$ is by the equation

$$N'(a) = \langle \bar{a}a \rangle_0. \tag{2.1.20}$$

These norms can be associated, for a multivector $a_{[k]} \in \Lambda_k(\mathbb{R}^{p,q})$, by the equation

$$N'(a_{[k]}) = (-1)^k N(a_{[k]}). \tag{2.1.21}$$

Proposition 2.4. *The applications $N, N' : \Gamma_{p,q} \to \mathbb{R}^*$ are homomorphisms.*

Proof. Indeed, let $a \in \Gamma_{p,q}$. Then, there are $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^{p,q}$ such that $a = \mathbf{v}_1 \ldots \mathbf{v}_k$.

Therefore,

$$\begin{aligned} N(a) &= \langle \mathbf{v}_k \dots \mathbf{v}_1 \mathbf{v}_1 \dots \mathbf{v}_k \rangle_0 \\ &= \mathbf{v}_k \dots \mathbf{v}_1 \mathbf{v}_1 \dots \mathbf{v}_k = N(\mathbf{v}_1) \dots N(\mathbf{v}_k). \end{aligned} \tag{2.1.22}$$

Then, for $a = \mathbf{v}_1 \dots \mathbf{v}_k$ and $b = \mathbf{u}_1 \dots \mathbf{u}_l$,

$$\begin{aligned} N(ab) &= \langle \widetilde{(ab)} ab \rangle_0 = \langle \widetilde{b} \widetilde{a} ab \rangle_0 \\ &= \langle \mathbf{u}_l \dots \mathbf{u}_1 \mathbf{v}_k \dots \mathbf{v}_1 \mathbf{v}_1 \dots \mathbf{v}_k \mathbf{u}_1 \dots \mathbf{u}_k \rangle_0 \\ &= N(\mathbf{v}_1) \dots N(\mathbf{v}_k) N(\mathbf{u}_1) \dots N(\mathbf{u}_l), \end{aligned} \tag{2.1.23}$$

which means that

$$N(ab) = N(a)N(b). \tag{2.1.24}$$

The result for N' follows analogously. □

The Pin Group

The group $\mathrm{Pin}(p,q)$ is defined as

$$\mathrm{Pin}(p,q) = \{a \in \Gamma_{p,q} : N(a) = \pm 1\}. \tag{2.1.25}$$

From this definition, it follows that[1]

$$\ker(\hat{\sigma}\big|_{\mathrm{Pin}(p,q)}) = \{\pm 1\} \simeq \mathbb{Z}_2. \tag{2.1.26}$$

One can similarly define the group $\mathrm{Pin}'(p,q)$ with respect to N' by

$$\mathrm{Pin}'(p,q) = \{a \in \Gamma_{p,q} : N'(a) = \pm 1\}. \tag{2.1.27}$$

If $a \in \mathcal{C}\ell^+_{p,q}$, then $N(a) = N'(a)$. Consequently,

$$\mathrm{Pin}(p,q) \cap \mathcal{C}\ell^+_{p,q} = \mathrm{Pin}'(p,q) \cap \mathcal{C}\ell^+_{p,q}. \tag{2.1.28}$$

The Spin Group

The group $\mathrm{Spin}(p,q)$ is defined as

$$\mathrm{Spin}(p,q) = \{a \in \Gamma^+_{p,q} : N(a) = \pm 1\}, \tag{2.1.29}$$

[1] $\mathbb{Z}_2$ is seen here as the multiplicative group $\mathbb{Z}_2 = \{\pm 1\}$.

where $\Gamma_{p,q}^+ = \Gamma_{p,q} \cap \mathcal{C}\ell_{p,q}^+$. Once again:

$$\ker(\hat{\sigma}\big|_{\mathrm{Spin}(p,q)}) = \{\pm 1\} \simeq \mathbb{Z}_2. \tag{2.1.30}$$

It is straightforward to notice that

$$\mathrm{Spin}(p,q) = \mathrm{Pin}(p,q) \cap \mathcal{C}\ell_{p,q}^+. \tag{2.1.31}$$

The definition of $\mathrm{Spin}'(p,q)$ using N' is clearly redundant since

$$\mathrm{Spin}'(p,q) = \mathrm{Pin}'(p,q) \cap \mathcal{C}\ell_{p,q}^+ = \mathrm{Pin}(p,q) \cap \mathcal{C}\ell_{p,q}^+ = \mathrm{Spin}(p,q). \tag{2.1.32}$$

Theorem 2.5.

$$\begin{aligned} Pin(p,q)/\mathbb{Z}_2 &\simeq O(p,q), \\ Spin(p,q)/\mathbb{Z}_2 &\simeq SO(p,q). \end{aligned} \tag{2.1.33}$$

The Reduced Pin and Spin Groups

One can now define the subgroups $\mathrm{Pin}_+(p,q)$ and Pin'_+ as

$$\mathrm{Pin}_+(p,q) = \{a \in \Gamma_{p,q} : N(a) = 1\} \tag{2.1.34}$$

and

$$\mathrm{Pin}'_+(p,q) = \{a \in \Gamma_{p,q} : N'(a) = 1\}. \tag{2.1.35}$$

Similarly, one can define $\mathrm{Spin}_+(p,q)$ as

$$\mathrm{Spin}_+(p,q) = \{a \in \Gamma_{p,q}^+ : N(a) = 1\}. \tag{2.1.36}$$

Notice that

$$\mathrm{Spin}_+(p,q) = \mathrm{Pin}_+(p,q) \cap \mathrm{Pin}'_+(p,q). \tag{2.1.37}$$

Finally, one can prove the following isomorphisms, as seen in ref. [5].

Theorem 2.6.

$$\begin{aligned} Pin_+(p,q)/\mathbb{Z}_2 &\simeq O_+(p,q), \\ Pin'_+(p,q)/\mathbb{Z}_2 &\simeq O^{\uparrow}(p,q), \\ Spin_+(p,q)/\mathbb{Z}_2 &\simeq SO_+(p,q). \end{aligned} \tag{2.1.38}$$

2.2 Algebraic Spinors

There are different known definitions of spinors in the literature. As we shall see, the group $\mathrm{Spin}_+(p,q)$ can be used to define the notion of *classical* spinors in $\mathcal{C}\ell_{p,q}$. However, we first present another definition of spinors that will be proven helpful to establish the former cited, which are called the *algebraic* spinors. This definition is intrinsically related to the concept of irreducible representations.

Definition 2.7. *A minimal left ideal of an algebra $\mathcal{A}$ is a left ideal S such that if $S' \subset S$ is a proper ideal of S, then $S' = \{0\}$.*

Definition 2.8. *An algebra $\mathcal{A}$ is called a simple algebra if every bilateral ideal is trivial and semisimple if it is the sum of simple algebras.*

It can be shown that every simple algebra is isomorphic to $\mathcal{M}(n,\mathbb{K})$, where $\mathbb{K}$ is a division algebra[2] and $\mathcal{M}(n,\mathbb{K})$ is the set of $n\times n$ matrices with entries in $\mathbb{K}$. Conversely, it is also possible to show that every algebra $\mathcal{M}(n,\mathbb{K})$ is indeed simple. This discussion can be found in Appendix G.

Let $\mathcal{A}$ be an unital algebra, where there is a natural underlying vector space structure. Also, consider the set of endomorphisms $\mathrm{End}(\mathcal{A})$ in order to construct the representations of $\mathcal{A}$. One can define a representation $\mathcal{L} : \mathcal{A} \to \mathrm{End}(\mathcal{A})$ as

$$\mathcal{L}(a)b = ab,\ \forall b \in \mathcal{A}. \tag{2.2.1}$$

It follows immediately that $\mathcal{L}(1) = 1$ and $\mathcal{L}(ab) = \mathcal{L}(a)\mathcal{L}(b)$, which implies that $\mathcal{L}$ is indeed a representation, called the *regular* representation. It is straightforward that this representation is faithful: if $\mathcal{L}(a)c = \mathcal{L}(b)c$, then $\mathcal{L}(a-b)c = (a-b)c = 0$. We can set $c = 1$, the unity in $\mathcal{A}$ and thus $a = b$, which means $\mathcal{L}$ is injective. It follows from the linearity of $\mathcal{L}$ that

$$\ker\mathcal{L} = \{0\}. \tag{2.2.2}$$

The underlying idea to the algebraic spinor definition is to find a subspace $S \subset \mathcal{A}$ such that $\mathcal{L} : \mathcal{A} \to \mathrm{End}(S)$ is an irreducible representation. It turns out that since we want $\mathcal{L}(a)s = as \in S$ for every $a \in A$ and that S is the minimal subspace such that this property holds (from the irreducibility), it follows that S must be a minimal left ideal in $\mathcal{A}$.

[2] For the purposes of this work, the division algebras will be $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, but the octonion algebra $\mathbb{O}$ is also a division algebra, although non-associative . See ref. [8] for more details.

Definition 2.9. *An element of a minimal left ideal of a Clifford algebra $\mathcal{C}\ell(V,g)$ is called an algebraic spinor if $\mathcal{C}\ell(V,g)$ is a simple algebra and a algebraic semispinor if $\mathcal{C}\ell(V,g)$ is semisimple.*

By inspecting the classification for Clifford algebras, it is straightforward to notice that every real Clifford algebra is either a simple or semisimple algebra. If $\mathcal{C}\ell(V,g) \simeq \mathcal{M}(n,\mathbb{K})$, then a minimal left ideal in $\mathcal{C}\ell(V,g)$ is isomorphic to $\mathbb{K}^n$, whereas in the case $\mathcal{C}\ell(V,g) \simeq \mathcal{M}(n,\mathbb{K}) \oplus \mathcal{M}(n,\mathbb{K})$, such ideal is isomorphic to $\mathbb{K}^n \oplus \mathbb{K}^n$. These results can also be found in Appendix G.

According to the classification for Clifford algebras, one can then divide the algebraic spinors for $\mathcal{C}\ell_{p,q}$ in the following classes:

- $p-q = 0,2 \mod 8$ | In this case, we have $\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{[n/2]},\mathbb{R})$, which implies by the observation above that an algebraic spinor is an element of a minimal left ideal isomorphic to $\mathbb{R}^{2^{[n/2]}}$.
- $p-q = 4,6 \mod 8$ | Here, $\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{[n/2]-1},\mathbb{H})$. Therefore, an algebraic spinor is in a space isomorphic to $\mathbb{H}^{2^{[n/2]-1}}$.
- $p-q = 3,7 \mod 8$ | Here we have $\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{[n/2]},\mathbb{C})$, which implies that the algebraic spinor space is isomorphic to $\mathbb{C}^{2^{[n/2]}}$.
- $p-q = 5 \mod 8$ | Here, the Clifford algebra is semisimple. We have the isomorphism $\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{[n/2]-1},\mathbb{H}) \oplus \mathcal{M}(2^{[n/2]-1},\mathbb{H})$, which yields the isomorphism from the semispinor space to $\mathbb{H}^{2^{[n/2]-1}}$. Clearly, the spinor space is then isomorphic to $\mathbb{H}^{2^{[n/2]-1}} \oplus \mathbb{H}^{2^{[n/2]-1}}$.
- $p-q = 1 \mod 8$ | This is also the case where the Clifford algebra is semisimple: $\mathcal{C}\ell_{p,q} \simeq \mathcal{M}(2^{[n/2]},\mathbb{R}) \oplus \mathcal{M}(2^{[n/2]},\mathbb{R})$. Therefore, the semispinor space is isomorphic to $\mathbb{R}^{2^{[n/2]}}$ and the spinor space is isomorphic to the direct sum $\mathbb{R}^{2^{[n/2]}} \oplus \mathbb{R}^{2^{[n/2]}}$.

$p-q \mod 8$	0	1	2	3
$S^A_{p,q}$	$\mathbb{R}^{2^{[n/2]}}$	$\mathbb{R}^{2^{[n/2]}} \oplus \mathbb{R}^{2^{[n/2]}}$	$\mathbb{R}^{2^{[n/2]}}$	$\mathbb{C}^{2^{[n/2]}}$
$p-q \mod 8$	4	5	6	7
$S^A_{p,q}$	$\mathbb{H}^{2^{[n/2]-1}}$	$\mathbb{H}^{2^{[n/2]-1}} \oplus \mathbb{H}^{2^{[n/2]-1}}$	$\mathbb{H}^{2^{[n/2]-1}}$	$\mathbb{C}^{2^{[n/2]}}$

Table 2.1: Algebraic Spinors Classification - [k] denotes the integer part of k.

The same analysis can be made upon the complexified case, which only depends on the parity of $n = p+q$.

$n=2k$	$S^A_{p,q} \simeq \mathbb{C}^{2^k}$
$n=2k+1$	$S^A_{p,q} \simeq \mathbb{C}^{2^k} \oplus \mathbb{C}^{2^k}$

Table 2.2: Algebraic Spinors Classification - Complexified Case

2.3 Classical Spinors

This definition of spinors is intrinsically connected with the Spin group, which was studied in detail previously. We recall here that

$$\mathrm{Spin}_+(p,q) = \{a \in \Gamma^+_{p,q} : N(a) = 1\},$$

where $\Gamma^+_{p,q}$ is the set of even elements in the Clifford-Lipschitz group.

Definition 2.10. *Let $\mathcal{C}\ell_{p,q}$ be the Clifford algebra of the quadratic space $\mathbb{R}^{p,q}$ and Spin$_+(p,q)$ the reduced Spin group associated to $\mathcal{C}\ell_{p,q}$. An element in the irreducible representation space of Spin$_+(p,q)$ is called a classical spinor.*

The following discussion consists of seeking these representation spaces. It is clear that $\mathrm{Spin}_+(p,q)$ is a subset of the even Clifford subalgebra. Consequently, we look for irreducible representations of such subalgebra, which will descend naturally to $\mathrm{Spin}_+(p,q)$. We can use Lemma 1.14, which shows us that a classical spinor in a quadratic space $\mathbb{R}^{p,q}$ is an algebraic spinor in the quadratic space $\mathbb{R}^{q,p-1}$ in order to do that. One can then construct the following table, which will assist us to classify these spinors accordingly[3].

$p-q \mod 8$	0	1	2	3
$\mathcal{C}\ell^+_{p,q}$	$\mathcal{M}(2^{[\kappa]},\mathbb{R}) \oplus \mathcal{M}(2^{[\kappa]},\mathbb{R})$	$\mathcal{M}(2^{[\kappa]},\mathbb{R})$	$\mathcal{M}(2^{[\kappa]},\mathbb{C})$	$\mathcal{M}(2^{[\kappa]-1},\mathbb{H})$
$p-q \mod 8$	4	5	6	7
$\mathcal{C}\ell^+_{p,q}$	$\mathcal{M}(2^{[\kappa]-1},\mathbb{H}) \oplus \mathcal{M}(2^{[\kappa]-1},\mathbb{H})$	$\mathcal{M}(2^{[\kappa]-1},\mathbb{H})$	$\mathcal{M}(2^{[\kappa]},\mathbb{C})$	$\mathcal{M}(2^{[\kappa]},\mathbb{R})$

Table 2.3: Real Even Subalgebras Classification

With that set, we can begin to classify the classical spinors. We use here that $\mathcal{C}\ell^+_{p,q} \simeq \mathcal{C}\ell_{p,q-1} = \mathcal{C}\ell_{r,s}$ and compare $r-s$ to find the real Clifford algebra isomorphism for $\mathcal{C}\ell^+_{p,q}$.

- $p-q = 1,7 \mod 8$ | In this case, $r-s = 0,2 \mod 8$. Therefore, we have $\mathcal{C}\ell^+_{p,q} \simeq \mathcal{M}(2^{[(n-1)/2]},\mathbb{R})$, where $n = p+q$. Then, a classical spinor is an element in a space isomorphic to $\mathbb{R}^{2^{[(n-1)/2]}}$.

[3]In the table, $\kappa = (n-1)/2$.

- $p-q = 2,6 \mod 8$ | Here, $r-s = 3,7 \mod 8$, and therefore $\mathcal{C}\ell^+_{p,q} \simeq \mathcal{M}(2^{[(n-1)/2]}, \mathbb{C})$. Thus, a classical spinor in this case is an element in a space isomorphic to $\mathbb{C}^{2^{[(n-1)/2]}}$.

- $p-q = 3,5 \mod 8$ | We have $r-s = 4,6 \mod 8$, which yields the isomorphism $\mathcal{C}\ell^+_{p,q} \simeq \mathcal{M}(2^{[(n-1)/2]-1}, \mathbb{H})$. The classical spinor space is then isomorphic to $\mathbb{H}^{2^{[(n-1)/2]-1}}$.

- $p-q = 4 \mod 8$ | Here the relation $r-s = 5 \mod 8$ shows that the even subalgebra is semisimple, wherein $\mathcal{C}\ell^+_{p,q} \simeq \mathcal{M}(2^{[(n-1)/2]-1}, \mathbb{H}) \oplus \mathcal{M}(2^{[(n-1)/2]-1}, \mathbb{H})$. Therefore, we have two nonequivalent representations of $\text{Spin}_+(p,q)$. We can decompose this space into "positive" and "negative" parts, where an element of each part is a (positive or negative) classical spinor. In that case, each of these elements is in a space isomorphic to $\mathbb{H}^{2^{[(n-1)/2]-1}}$.

- $p-q = 0 \mod 8$ | Finally, we have $r-s = 1 \mod 8$. This setup also corresponds to a semisimple algebra, namely $\mathcal{C}\ell^+_{p,q} \simeq \mathcal{M}(2^{[(n-1)/2]}, \mathbb{R}) \oplus \mathcal{M}(2^{[(n-1)/2]}, \mathbb{R})$ and, again, we are able to split the space in two, the positive and negative classical spinor sets. Thereafter, it follows these semispinor are elements in a space isomorphic to $\mathbb{R}^{2^{[(n-1)/2]}}$.

$p-q \mod 8$	0	1	2	3
$S^C_{p,q}$	$\mathbb{R}^{2^{[(n-1)/2]}} \oplus \mathbb{R}^{2^{[(n-1)/2]}}$	$\mathbb{R}^{2^{[(n-1)/2]}}$	$\mathbb{C}^{2^{[(n-1)/2]}}$	$\mathbb{H}^{2^{[(n-1)/2]-1}}$
$p-q \mod 8$	4	5	6	7
$S^C_{p,q}$	$\mathbb{H}^{2^{[(n-1)/2]-1}} \oplus \mathbb{H}^{2^{[(n-1)/2]-1}}$	$\mathbb{H}^{2^{[(n-1)/2]-1}}$	$\mathbb{C}^{2^{[(n-1)/2]}}$	$\mathbb{R}^{2^{[(n-1)/2]}}$

Table 2.4: Classical Spinors Classification

For the case involving complex Clifford algebras, by the isomorphism $\mathcal{C}\ell^+_{p,q} \simeq \mathcal{C}\ell_{p,q-1}$ it follows that

$$\mathcal{C}\ell^+_{\mathbb{C}}(n) \simeq \mathcal{C}\ell_{\mathbb{C}}(n-1). \tag{2.3.1}$$

$n = 2k$	$\mathbb{C}^{2^{k-1}} \oplus \mathbb{C}^{2^{k-1}}$
$n = 2k+1$	$\mathbb{C}^{2^k}$

Table 2.5: Classical Spinors Classification - Complexified Case

2.4 The Dirac Spinor

As an example, one can analyze the so-called Dirac spinor. Consider the group of 2×2 complex matrices with unitary determinant, namely

$$\mathrm{SL}(2,\mathbb{C}) = \{A \in M(2,\mathbb{C}) : \det(A) = 1\}. \tag{2.4.1}$$

It follows that

$$\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \simeq \mathrm{SO}_+(1,3), \tag{2.4.2}$$

and one says that $\mathrm{SL}(2,\mathbb{C})$ is the *double cover* of $\mathrm{SO}_+(1,3)$. More details can be found in Appendix H. There are two non-equivalent representations of $\mathrm{SL}(2,\mathbb{C})$, denoted $D^{(1/2,0)}$ and $D^{(0,1/2)}$, the left-handed and right-handed representations, respectively. The elements of the representation space associated with them are called the Weyl spinors, which are elements in $\mathbb{C}^2$. Dirac spinors, concerned in the next section, take into account reducible representations in the form $D^{(1/2,0)} \oplus D^{(0,1/2)}$, that is, elements $\psi \in \mathbb{C}^4$. In fact, as we shall see, Dirac spinors are classical spinors[4] of the complexified space $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$. Some physical realizations are taken into account and we discuss the so-called bilinear covariants associated to such spinors, as well as identities involving them and a reconstruction theorem.

As seen in [9], wave functions are described in non-relativistic quantum mechanics by the time dependent Schrödinger equation:

$$\frac{-\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{2.4.3}$$

for some potential V. In fact, this is an energy conservation equation: the kinect energy $\frac{p^2}{2m}$ plus the potential energy V equals the total energy E, where p and E are replaced by their equivalent operators

$$p = -i\hbar\nabla \quad \text{and} \quad E = -i\hbar\frac{\partial}{\partial t}. \tag{2.4.4}$$

In relativistic quantum mechanics, the energy equation is

$$\frac{E^2}{c^2} - p^2 = m^2c^2.$$

[4]The algebraic spinors space for this Clifford algebra is isomorphic to $\mathbb{C}^4$ and hence there is a natural way to relate the two different spinor definition in $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$. More details can be found in ref. [3].

Inserting the operators for p and E, it yields the Klein-Gordon equation

$$E^2 - p^2 - m^2 = 0 \mapsto \hbar^2\left(\frac{-\partial^2}{c^2\partial t^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\psi(x) - m^2c^2 = 0. \tag{2.4.5}$$

Dirac then linearized the above equation, resulting in

$$i\hbar\left(\gamma_0\frac{1}{c}\frac{\partial}{\partial t} + \gamma_1\frac{\partial}{\partial x_1} + \gamma_2\frac{\partial}{\partial x_2} + \gamma_3\frac{\partial}{\partial x_3}\right)\psi(x) = mc\psi(x), \tag{2.4.6}$$

as long as the symbols γ_μ satisfied the following properties:

$$\begin{aligned} &\gamma_0^2 = I, \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -I, \\ &\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu, \quad \text{for } \mu \neq \nu. \end{aligned} \tag{2.4.7}$$

A set of matrices that fulfill these relations is

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.4.8}$$
$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Setting $x_0 = ct$, one can write the condensed form of Dirac equation as

$$i\hbar\gamma^\mu\partial_\mu\psi = mc\psi. \tag{2.4.9}$$

Considering the interaction with an electromagnetic field $F^{\mu\nu}$, one can write the equation introducing such field via the potential $(\frac{1}{c}V, A_x, A_y, A_z)$ of $F^{\mu\nu}$. It leads to the following equation:

$$\gamma_\mu(i\hbar\partial^\mu - eA^\mu)\psi = mc\psi, \tag{2.4.10}$$

where the wave function is a column spinor with complex entries, that is, for some

$\psi_1, \ldots, \psi_4 : \mathbb{R}^{1,3} \to \mathbb{C}$ we have

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \in \mathbb{C}^4. \tag{2.4.11}$$

2.4.1 Bilinear Covariants

The Dirac adjoint of $\psi \in \mathbb{C}^4$ is a row matrix

$$\psi^\dagger \gamma_0 = \begin{pmatrix} \psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^* \end{pmatrix}, \tag{2.4.12}$$

in which ψ_i^* is the complex conjugate of ψ_i. The so-called observables represent the physical state of the electron, which is described by the Dirac spinor in the above presented formalism. For a spinor $\psi \in \mathbb{C}^4$, these are determined by the following bilinear covariants [10, 11, 12]:

$$\Omega_1 = \psi^\dagger \gamma_0 \psi = \psi_1^* \psi_1 + \psi_2^* \psi_2 - \psi_3^* \psi_3 - \psi_4^* \psi_4, \tag{2.4.13}$$
$$J_\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi, \tag{2.4.14}$$
$$S_{\mu\nu} = \frac{1}{2} \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi, \tag{2.4.15}$$
$$K_\mu = \psi^\dagger \gamma_0 i \gamma_{0123} \gamma_\mu \psi, \tag{2.4.16}$$
$$\Omega_2 = \psi^\dagger \gamma_0 \gamma^{0123} \psi; \qquad \gamma^{0123} = -\gamma_0 \gamma_1 \gamma_2 \gamma_3. \tag{2.4.17}$$

Their integrals over a space-like domain give expectation values of physical observables: notice that $J^0 = \psi^\dagger \psi$ integrated over a space domain gives the probability of finding the electron in that domain. Furthermore, $J_k = \psi^\dagger \gamma_0 \gamma_k \psi$, for $k = 1, 2, 3$, gives the current of probability

$$\vec{\mathbf{J}} = J_\mu \gamma^\mu, \tag{2.4.18}$$

which satisfies the continuity equation

$$\frac{1}{c} \frac{\partial J^0}{\partial t} + \frac{\partial J^k}{\partial x^k} = 0. \tag{2.4.19}$$

The bivector $\mathbf{S} = S_{\mu\nu} \gamma^{\mu\nu}$ is interpreted as the electromagnetic moment density; the probability density of the electromagnetic moment of the electron. The vector $\mathbf{K} = K_\mu \gamma^\mu$ gives the direction of the spin of the electron. The first and last equations can be unified into a single equation given by

$$\Omega = \Omega_1 + \Omega_2 \gamma_{0123}. \tag{2.4.20}$$

2.4.2 Fierz Identities and Reconstruction

Fierz identities are quadratic relations between the bilinear covariants of a Dirac spinor. The identities are given by the expressions

$$\mathbf{J}^2 = \Omega_1^2 + \Omega_2^2, \tag{2.4.21}$$

$$\mathbf{K}^2 = -\mathbf{J}^2, \tag{2.4.22}$$

$$\mathbf{J} \cdot \mathbf{K} = 0, \tag{2.4.23}$$

$$\mathbf{J} \wedge \mathbf{K} = -(\Omega_2 + \Omega_1 \gamma_{0123})\mathbf{S}. \tag{2.4.24}$$

It is possible to show these identities by direct computation, and the first two are shown in Appendix I.

One can use the bilinear covariants to recover the original Dirac spinor up to a phase [10, 11, 12]. The Fierz identities are sufficient to examine the case where $\Omega_1 = \psi^\dagger \gamma_0 \psi \neq 0$ or $\Omega_2 = \psi^\dagger \gamma_0 \gamma_{0123} \psi \neq 0$. However, they are insufficient for the case $\Omega = 0$ [13, 14, 15]. We follow to present such results, as seen in the aforesaid references.

Consider a spinor ψ and let Ω_1 $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$ and Ω_2 be its bilinear covariants. Moreover, let τ be a spinor such that $\widetilde{\tau}^* \psi \neq 0$ in $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$. Then, the spinor ψ is proportional to $Z\tau$, where Z is called an *aggregate* and is given by

$$Z = \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \Omega_2 \gamma_{0123}. \tag{2.4.25}$$

This means that ψ and $Z\tau$ differ by a complex factor. Using the algorithm as proposed in [3, 12, 13, 16], one can recover the spinor by making

$$\psi = \frac{1}{4N} e^{-i\alpha} Z\tau, \tag{2.4.26}$$

$$N = \frac{1}{2}\sqrt{\tau^\dagger \gamma_0 Z \tau}. \tag{2.4.27}$$

Conversely, if ψ is given, it is possible to write it as in Eq.(2.4.26) by letting

$$N = |\tau^\dagger \gamma_0 \psi|, \tag{2.4.28}$$

$$e^{-i\alpha} = \frac{1}{N}\tau^\dagger\gamma_0\psi. \tag{2.4.29}$$

To conclude, a spinor τ such that $\widetilde{\tau}^*\psi \neq 0$ works as a projector of the aggregate Z, extracting the relevant parts of Z to the reconstructed spinor ψ. Therefore, a spinor ψ is described by its bilinear covariants up to a phase-factor, with the support of the aggregate $Z = \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \Omega_2\gamma_{0123}$.

2.4.3 Boomerangs

We shall now discuss in what cases arbitrary multivectors are actually bilinear covariants for a spinor $\psi \in \mathbb{C}\otimes\mathcal{C}\ell_{1,3}$.

Definition 2.11. *Let Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$ and Ω_2 be arbitrary multivectors. If they satisfy the Fierz identities, then their aggregate $Z = \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{01230} + \Omega_2\gamma_{0123}$ is called a Fierz aggregate.*

Definition 2.12. *A multivector $Z = \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \Omega_2\gamma_{0123}$ such that $\tilde{Z}^* = Z$ is called a boomerang if its components Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$ and Ω_2 are bilinear covariants for a spinor $\psi \in \mathbb{C}^4$.*

In the non-null case, as well as in the null case $\Omega = 0$, a spinor ψ is determined up to a phase factor by its aggregate of bilinear covariants. Similarly, Z can be determined by the original spinor via the formula $Z = 4\psi\tilde{\psi}^* = 4\psi\psi^\dagger\gamma_0$, which means it *"boomerangs"* back. Fixing Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$ and Ω_2 as arbitrary multivectors, then in the non-singular case $\Omega \neq 0$ their aggregate can be factored as [3, 13]:

$$Z = (\Omega_1 + \mathbf{J} + \Omega_2\gamma_{0123})(1 + i(\Omega_1 + \Omega_2\gamma_{0123})^{-1}\mathbf{K}\gamma_{0123}). \tag{2.4.30}$$

Using this factorization, it can be proved that if the multivectors satisfy the Fierz identities, and provided that $J^0 > 0$ along with $4\langle\widetilde{\tau}^* Z\tau\rangle_0 = \tau^\dagger\gamma_0 Z\tau > 0$ for all non-zero spinors τ, then Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$ and Ω_2 are bilinear covariants for some spinor ψ such that

$$\psi = \frac{1}{4N}Z\tau, \quad N = \frac{1}{2}\sqrt{\tau^\dagger\gamma_0 Z\tau}, \tag{2.4.31}$$

where two such spinors differ only in their phases. Furthermore, one can prove that in the non-null case $\Omega \neq 0$, Z is a boomerang if and only if Z is a Fierz aggregate. Nevertheless, in the case $\Omega = 0$ there are aggregates Z which are Fierz aggregates but still do not boomerang, for instance

$$Z = \mathbf{J}, \quad \mathbf{J}^2 = 0, \quad \mathbf{J} \neq 0. \tag{2.4.32}$$

If $\Omega = 0$, $\mathbf{J}$, $\mathbf{S}$ and $\mathbf{K}$ satisfy the Fierz identities, then for a spinor constructed by

$$\psi = \frac{1}{4N} Z\tau, \quad \text{where} \quad Z = \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} \tag{2.4.33}$$

in general we have $Z \neq 4\psi\tilde{\psi}^*$. To guarantee that Z boomerangs in this case, one could replace the Fierz identities by the more restrictive conditions

$$Z^2 = 4\Omega_1 Z, \quad Z\gamma_\mu Z = 4J_\mu Z, \quad Zi\gamma\mu\nu Z = 4S_{\mu\nu}Z, \tag{2.4.34}$$

$$Zi\gamma_{0123}\gamma_\mu Z = 4K_\mu Z, \quad Z\gamma_{0123}Z = -4\Omega_2 Z. \tag{2.4.35}$$

That way, if $Z = \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123}$ is a boomerang, then $Z^2 = 0$ and each dimension degree vanishes, which implies that

$$\langle Z^2\rangle_0 = \mathbf{J}^2 - \mathbf{S}\rfloor\mathbf{S} - \mathbf{K}^2 \tag{2.4.36}$$

$$\langle Z^2\rangle_1 = 2\gamma_{0123}(\mathbf{S}\wedge\mathbf{K}) \qquad \mathbf{K} \text{ in the plane of } \mathbf{S} \tag{2.4.37}$$

$$\langle Z^2\rangle_2 = i2\gamma_{0123}(\mathbf{J}\wedge\mathbf{K}) \qquad \mathbf{J} \text{ and } \mathbf{K} \text{ are parallel} \tag{2.4.38}$$

$$\langle Z^2\rangle_3 = i2\mathbf{J}\wedge\mathbf{S} \qquad \mathbf{J} \text{ in the plane of } \mathbf{S} \tag{2.4.39}$$

$$\langle Z^2\rangle_4 = -\mathbf{S}\wedge\mathbf{S} \qquad \mathbf{S} \text{ is simple} \tag{2.4.40}$$

Altogether, one must have that

$$Z = \mathbf{J}(1 + i\mathbf{s} + ih\gamma_{0123}), \tag{2.4.41}$$

where h is a real number and $\mathbf{s}$ is a vector orthogonal to $\mathbf{J}$ such that $\mathbf{s}^2 \geq \mathbf{0}$. One can prove that Z is a boomerang if both conditions are satisfied simultaneously.

Theorem 2.13. *Let Z be an aggregate such that $\tau^\dagger\gamma_0 Z\tau \geq 0$ for all spinors τ and that $J^0 > 0$. Then the following statements hold:*

1. *Z is a boomerang if and only if $Z\gamma_0\tilde{Z}^* = 4J^0 Z$.*
2. *If $\Omega \neq 0$, Z is a boomerang if and only if it is a Fierz aggregate.*
3. *If $\Omega = 0$, Z is a boomerang if and only if $Z = \mathbf{J}(1 + i\mathbf{s} + ih\gamma_{0123})$ where $\mathbf{J}^2 = 0$, $\mathbf{s}$ is a vector orthogonal to $\mathbf{J}$ such that $\mathbf{s}^2 \geq 0$ and h is a real number such that $h = \pm\sqrt{1+\mathbf{s}^2}$ and $|h| \leq 1$.*

Therefore, since it is possible to characterize spinors in $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ by their bilinear covariants, up to a phase, Lounesto classified spinor fields into six disjoint classes, wherein $\mathbf{J} \neq 0$ [3]:

$$
\begin{aligned}
&1) \qquad \Omega_1(x) \neq 0, \quad \Omega_2(x) \neq 0, \quad \mathbf{S}(x) \neq 0, \quad \mathbf{K}(x) \neq 0, && (2.4.42a)\\
&2) \qquad \Omega_1(x) \neq 0, \quad \Omega_2(x) = 0, \quad \mathbf{S}(x) \neq 0, \quad \mathbf{K}(x) \neq 0, && (2.4.42b)\\
&3) \qquad \Omega_1(x) = 0, \quad \Omega_2(x) \neq 0, \quad \mathbf{S}(x) \neq 0, \quad \mathbf{K}(x) \neq 0, && (2.4.42c)\\
&4) \qquad \Omega_1(x) = 0, \quad \Omega_2(x) = 0, \quad \mathbf{S}(x) \neq 0, \quad \mathbf{K}(x) \neq 0, && (2.4.42d)\\
&5) \qquad \Omega_1(x) = 0, \quad \Omega_2(x) = 0, \quad \mathbf{S}(x) \neq 0, \quad \mathbf{K}(x) = 0, && (2.4.42e)\\
&6) \qquad \Omega_1(x) = 0, \quad \Omega_2(x) = 0, \quad \mathbf{K}(x) \neq 0, \quad \mathbf{S}(x) = 0. && (2.4.42f)
\end{aligned}
$$

Singular spinor fields of types-4, -5, and -6 are flag-dipoles, flagpoles and dipole spinor fields, respectively. Since **K** and **J** are 1-form fields, they can be identified as poles, as the 2-form field **S** is a flag. Hence, type-5 spinor fields, having a null pole $\mathbf{K} = 0$, the pole $\mathbf{J} \neq 0$ and $\mathbf{S} \neq 0$ are called flag-poles. Type-4 spinor fields have two poles $\mathbf{J} \neq 0$ and $\mathbf{K} \neq 0$ and the flag $\mathbf{S} \neq 0$, being denominated a flag-dipole. Regarding type-6 spinors, still $\mathbf{J} \neq 0$ and $\mathbf{K} \neq 0$, however the flag **S** is zero, terming it a dipole spinor field. Flag-dipole spinor fields were shown to be a legitimate solution of the Dirac field equation in a torsional setup [17, 18], whereas Elko and Majorana uncharged spinor fields represent type-5 spinors, although a recent example of a charged flagpole spinor has been shown to be a solution of the Dirac equation [19]. More physical important examples on the Lounesto's classification can be found in refs. [20, 21].

3 The Triality Principle

In this chapter we shall present the triality principle, as constructed in [5, 22]. We discuss spinor inner products, which can be defined with respect to two different involutions. Then, a brief introduction to the principle is given, leading to a direct construction. The omitted proofs can be found in the aforesaid references.

3.1 The Spinor Inner Product

We shall consider in the forthcoming subsection that all spinor spaces are defined according to the algebraic definition. One can endow such space with an inner product, with the adjoint property corresponding to an anti-automorphism in the Clifford algebra. It is well known that there are two anti-automorphisms defined for $\mathcal{C}\ell_{p,q}$, which are the reversion and the conjugation. These two, therefore, induce two types of spinor inner products. Let S be the space of algebraic spinors in $\mathcal{C}\ell_{p,q} \simeq End_{\mathbb{K}}(S)$. One can define the spinor inner products $\tilde{h} : S \times S \to \mathbb{K}$ and $\overline{h} : S \times S \to \mathbb{K}$ by the relations

$$\tilde{h}(a\psi, \phi) = \tilde{h}(\psi, \tilde{a}\phi), \tag{3.1.1}$$

$$\overline{h}(a\psi, \phi) = \overline{h}(\psi, \overline{a}\phi), \tag{3.1.2}$$

where the adjoint corresponds to the reversion in the first equation and the conjugation in the second.

For the sake of generality, one can define ψ° to be either $\tilde{\psi}$ or $\overline{\psi}$, for $\psi \in \mathcal{C}\ell_{p,q}$. As seen before, $\mathcal{C}\ell_{p,q}$ is a simple (or semisimple) algebra and therefore, since S is a minimal left ideal, there is an idempotent f such that $S = \mathcal{C}\ell_{p,q}f = \mathbb{K}^n$, and similarly for the semispinor spaces, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Moreover, $f\mathcal{C}\ell_{p,q}f = \mathbb{K}$ and for $\psi, \phi \in S$, one has $\psi f = \psi$ and $\phi f = \phi$ (these results can be found in Appendix G). That way,

$$\psi^\circ \phi = (\psi f)^\circ \phi f = f^\circ \psi^\circ \phi f. \tag{3.1.3}$$

Consequently, one can notice that $h(\psi,\phi) = \psi^\circ\phi$ implies that

$$h(a\psi,\phi) = h(\psi, a^\circ\phi), \tag{3.1.4}$$

Therefore, if $f^\circ = f$, then $\psi^\circ\phi$ is a spinor scalar product, since $\psi^\circ\phi = f^\circ\psi^\circ\phi f \in \mathbb{K}$. If $f^\circ \neq f$, one can take an element $s \in \mathcal{C}\ell^*_{p,q}$ such that $sf^\circ s^{-1} = f$. Then, $s\psi^\circ\phi$ is a spinor scalar product. Namely, one can define

$$\tilde{h}(\psi,\phi) = s\tilde{\psi}\phi, \tag{3.1.5}$$

$$\overline{h}(\psi,\phi) = s\overline{\psi}\phi, \tag{3.1.6}$$

where $s \in \mathcal{C}\ell^*_{p,q}$. As a result of that,

$$\overline{h}(\psi,\phi) = s\overline{\psi}\phi = s\tilde{\hat{\psi}}\phi = \tilde{h}(\hat{\psi},\phi). \tag{3.1.7}$$

Notice that, indeed, the relations in Eq.(3.1.4) are preserved. Namely,

$$\overline{h}(a\psi,\phi) = s\overline{a\psi}\phi = s\overline{\psi}\overline{a}\phi = \overline{h}(\psi,\overline{a}\phi), \tag{3.1.8}$$

$$\tilde{h}(a\psi,\phi) = s\widetilde{a\psi}\phi = s\tilde{\psi}\tilde{a}\phi = \tilde{h}(\psi,\tilde{a}\phi). \tag{3.1.9}$$

3.2 The Triality Principle in Clifford Algebras

Suppose that an (algebraic) spinor space S can be written as $S = S^+ \oplus S^-$, where $S^\pm$ are the semispinor spaces. We would like to know if there is a quadratic space (V, g) such that the associated semispinor spaces $S^\pm$ would have the same dimension as V. Letting $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we look for $\dim V = \dim S^+ = \dim S^- = n$. Additionally, we need the index of the metric g to be the same as to the inner product h in $S^\pm$, which we take to be the spinor inner product associated with reversion. Looking at the representation table for algebraic spinors, one can see that this can only happen for $n = 1, 2, 4$ or 8. We study the case $n = 8$. We let $V = \mathbb{C}^8$ in general, but one could look at $V = \mathbb{R}^{0,8}, \mathbb{R}^{4,4}, \mathbb{R}^{8,0}$ as well, minding the index of the associated inner products. Let $E = V \oplus S^+ \oplus S^-$ be the 24 dimensional space and write $E \ni \phi = \mathbf{x} + \mathfrak{u} + \mathfrak{v}$, fixing $\mathbf{x} \in V$, $\mathfrak{u} \in S^+$ and $\mathfrak{v} \in S^-$.

Lemma 3.1. *If $\mathbf{x} \in V$ is such that $\mathbf{x}^2 \neq 0$, then for every $\mathfrak{v} \in S^-$ there is $\mathfrak{u} \in S^+$ such that $\mathfrak{v} = \mathbf{x}\mathfrak{u}$.*

The spinor metric $h : S^\pm \times S^\pm \to \mathbb{K}$ is defined in each one of the spaces $S^\pm$ as

$$h(\mathfrak{u}, \mathbf{x}\mathfrak{v}) = h(\tilde{\mathbf{x}}\mathfrak{u}, \mathfrak{v}) = \widehat{\mathbf{x}\mathfrak{u}}\mathfrak{v}, \tag{3.2.1}$$

where the dual adjoint is given by

$$\begin{aligned} (\widehat{\cdot}) : S^\pm \quad &\longrightarrow \quad (S^\pm)^* = \operatorname{Hom}(S^\pm, \mathbb{K}) \\ \psi \quad &\mapsto \quad \widehat{\psi}, \quad \text{where } \widehat{\psi}(\phi) = h(\phi, \psi), \ \forall \phi \in S^\pm. \end{aligned} \tag{3.2.2}$$

One can further define a symmetric bilinear form $\mathcal{B}$ endowing the vector space E, using the spinor inner product h and the metric g from V. Let $\phi_i = \mathbf{x}_i + \mathfrak{u}_i + \mathfrak{v}_i$ for $i \in \{1, 2, 3\}$ and define [4, 22, 23]

$$\mathcal{B}(\phi_1, \phi_2) = g(\mathbf{x}_1, \mathbf{x}_2) + h(\mathfrak{u}_1, \mathfrak{u}_2) + h(\mathfrak{v}_1, \mathfrak{v}_2). \tag{3.2.3}$$

Now, a totally symmetric trilinear tensor $T : E \times E \times E \to \mathbb{K}$ can be defined as

$$\begin{aligned} T(\phi_1, \phi_2, \phi_3) = &\, h(\mathfrak{u}_1, \mathbf{x}_2\mathfrak{v}_3) + h(\mathfrak{u}_1, \mathbf{x}_3\mathfrak{v}_2) + h(\mathfrak{u}_2, \mathbf{x}_3\mathfrak{v}_1) \\ &+ h(\mathfrak{u}_2, \mathbf{x}_1\mathfrak{v}_3) + h(\mathfrak{u}_3, \mathbf{x}_2\mathfrak{v}_1) + h(\mathfrak{u}_3, \mathbf{x}_1\mathfrak{v}_2). \end{aligned} \tag{3.2.4}$$

Having that, it is possible to endow the space E with a commutative and non-associative product $\circ : E \times E \to E$, the so-called Chevalley product. It is implicitly defined by

$$T(\phi_1, \phi_2, \phi_3) = \mathcal{B}(\phi_1 \circ \phi_2, \phi_3). \tag{3.2.5}$$

From the symmetry property of T, it follows that $\phi_1 \circ \phi_2 = \phi_2 \circ \phi_1$, that is, the product is commutative. On the other hand, let $\mathbf{x} \in V$ be a non-isotropic vector, $\mathfrak{u} \in S^+$ and $\mathfrak{v} \in S^-$. Notice that

$$\begin{aligned} \mathcal{B}(\mathbf{x} \circ \mathfrak{u}, \mathfrak{v}) = T(\mathbf{x}, \mathfrak{u}, \mathfrak{v}) = h(\mathfrak{u}, \mathbf{x}\mathfrak{v}) = h(\mathbf{x}\mathfrak{u}, \mathfrak{v}) = \mathcal{B}(\mathbf{x}\mathfrak{u}, \mathfrak{v}), \\ \mathcal{B}(\mathbf{x} \circ \mathfrak{v}, \mathfrak{u}) = T(\mathbf{x}, \mathfrak{v}, \mathfrak{u}) = h(\mathfrak{v}, \mathbf{x}\mathfrak{u}) = h(\mathbf{x}\mathfrak{v}, \mathfrak{u}) = \mathcal{B}(\mathbf{x}\mathfrak{v}, \mathfrak{u}). \end{aligned}$$

Therefore, $\mathbf{x} \circ \mathfrak{u} = \mathbf{x}\mathfrak{u}$ and $\mathbf{x} \circ \mathfrak{v} = \mathbf{x}\mathfrak{v}$. We have then $\mathbf{x} \circ \mathfrak{u} \in V^-$ and $\mathbf{x} \circ \mathfrak{v} \in V^+$ and by the same reasoning it follows that $\mathbf{x} \circ (\mathbf{x} \circ \mathfrak{u}) = \mathbf{x}\mathbf{x}\mathfrak{u} = g(\mathbf{x}, \mathbf{x})\mathfrak{u} \neq 0$. Nevertheless, it follows that

$$\mathcal{B}(\mathbf{x} \circ \mathbf{x}, \mathfrak{v}) = 0 = \mathcal{B}(0, \mathfrak{v}), \tag{3.2.6}$$

and therefore $\mathbf{x} \circ \mathbf{x} = 0$. In fact, if ψ and ϕ are in the same subspace in E, there holds $\psi \circ \phi = 0$. Thus, it is straightforward to see that $(\mathbf{x} \circ \mathbf{x}) \circ \mathfrak{u} \neq \mathbf{x} \circ (\mathbf{x} \circ \mathfrak{u})$, which means that the Chevalley product is not alternative and, in particular, non-associative. Now, the spinor norm of $\mathbf{x}\circ\mathfrak{u}$ can be derived from the norms of $\mathbf{x}$ and $\mathfrak{u}$ by the relation

$$h(\mathbf{x}\circ\mathfrak{u},\mathbf{x}\circ\mathfrak{u}) = h(\mathbf{x}\mathfrak{u},\mathbf{x}\mathfrak{u}) = (\widehat{\mathbf{x}\mathfrak{u}})\mathbf{x}\mathfrak{u} = \widehat{\mathfrak{u}}\,\widehat{\mathbf{x}}\mathbf{x}\mathfrak{u} = g(\mathbf{x},\mathbf{x})h(\mathfrak{u},\mathfrak{u}), \tag{3.2.7}$$

and one can prove that the following inclusions hold:

$$V\circ S^+\subset S^-, \qquad S^+\circ S^-\subset V, \qquad S^-\circ V\subset S^+. \tag{3.2.8}$$

Indeed, let $\mathbf{x}_1\in V$, $\mathfrak{u}_2\in S^+$ and $\mathbf{x}_3+\mathfrak{u}_3\in V\oplus S^+$ be non-null elements. Then, $T(\mathbf{x}_1,\mathfrak{u}_2,\mathbf{x}_3+\mathfrak{u}_3) = \mathcal{B}(\mathbf{x}_1\circ\mathfrak{u}_2,\mathbf{x}_3+\mathfrak{u}_3) = 0$ and therefore, and here we suppose h is also non-degenerate, it follows that $\mathbf{x}_1\circ\mathfrak{u}_2\in S^-$, which is the orthogonal space to $V\oplus S^+$. Similarly one can derive the other relations.

Let now σ be a representation of the Clifford-Lipschitz group in $S^\pm$ and χ the adjoint representation of the same group in V. Then, these representations induce a representation Y in E. Given an unitary element of the Clifford-Lipschitz group $s\in\Gamma_V^+$, the action of this representation as $Y(s):E\to E$ can be defined as

$$Y(s)(\mathbf{x}+\mathfrak{u}+\mathfrak{v}) = \chi(s)\mathbf{x}+\sigma(s)\mathfrak{u}+\sigma(s)\mathfrak{v} = s\mathbf{x}s^{-1}+s\mathfrak{u}+s\mathfrak{v}, \tag{3.2.9}$$

where this map is orthogonal with respect to $\mathcal{B}$. Indeed:

$$\begin{aligned}
&\mathcal{B}(Y(s)\phi_1,Y(s)\phi_2)\\
&\quad= \mathcal{B}(\chi(s)\mathbf{x}_1+\sigma(s)\mathfrak{u}_1+\sigma(s)\mathfrak{v}_1,\chi(s)\mathbf{x}_2+\sigma(s)\mathfrak{u}_2+\sigma(s)\mathfrak{v}_2)\\
&\quad= g(\chi(s)\mathbf{x}_1,\chi(s)\mathbf{x}_2)+h(\sigma(s)\mathfrak{u}_1,\sigma(s)\mathfrak{u}_2)+h(\sigma(s)\mathfrak{v}_1,\sigma(s)\mathfrak{v}_2)\\
&\quad= g(s\mathbf{x}_1s^{-1},s\mathbf{x}_2s^{-1})+h(s\mathfrak{u}_1,s\mathfrak{u}_2)+h(s\mathfrak{v}_1,s\mathfrak{v}_2)\\
&\quad= g(\mathbf{x}_1,\mathbf{x}_2)+h(\mathfrak{u}_1,\mathfrak{u}_2)+h(\mathfrak{v}_1,\mathfrak{v}_2)\\
&\quad= \mathcal{B}(\phi_1,\phi_2).
\end{aligned}$$

Now, notice that if $\mathbf{x}_0\in V$ is such that $\mathbf{x}_0^2=1$, then $Y^2(\mathbf{x}_0)=I$, the identity operator:

$$\begin{aligned}
Y^2(\mathbf{x}_0)(\mathbf{x}+\mathfrak{u}+\mathfrak{v}) &= Y(\mathbf{x}_0)(\mathbf{x}_0\mathbf{x}\mathbf{x}_0+\mathbf{x}_0\mathfrak{u}+\mathbf{x}_0\mathfrak{v})\\
&= \mathbf{x}_0^2\mathbf{x}\mathbf{x}_0^2+\mathbf{x}_0^2\mathfrak{u}+\mathbf{x}_0^2\mathfrak{v} = \mathbf{x}+\mathfrak{u}+\mathfrak{v}.
\end{aligned}$$

One can now begin the construction of the triality automorphism. Let $\mathfrak{u}_0\in S^+$ such that $h(\mathfrak{u}_0,\mathfrak{u}_0) = 1$ and define the linear mapping

$$\begin{aligned} \zeta : S^+ &\to \mathcal{L}(V, S^-) \\ \mathfrak{u}_0 &\mapsto \zeta(\mathfrak{u}_0) : V \to S^- \\ &\qquad\qquad \mathbf{x} \mapsto \zeta(\mathfrak{u}_0)(\mathbf{x}) := \mathbf{x} \circ \mathfrak{u}_0 \end{aligned} \tag{3.2.10}$$

where it follows that $\zeta(\mathfrak{u}_0)$ is orthogonal with respect to $\mathcal{B}$. Indeed, one can see that

$$\begin{aligned} h(\zeta(\mathfrak{u}_0)\mathbf{x}_1, \zeta(\mathfrak{u}_0)\mathbf{x}_2) &= h(\mathbf{x}_1 \circ \mathfrak{u}_0, \mathbf{x}_2 \circ \mathfrak{u}_0) = h(\mathbf{x}_1\mathfrak{u}_0, \mathbf{x}_2\mathfrak{u}_0) \\ &= \widehat{(\mathbf{x}_1\mathfrak{u}_0)}\mathbf{x}_2\mathfrak{u}_0 = \widehat{\mathfrak{u}_0}\widehat{\mathbf{x}_1}\mathbf{x}_2\mathfrak{u}_0 = g(\mathbf{x}_1, \mathbf{x}_2)h(\mathfrak{u}_0, \mathfrak{u}_0) = g(\mathbf{x}_1, \mathbf{x}_2), \end{aligned}$$

which relates the spinor norm of $\mathbf{x} \circ \mathfrak{u}_0 \in S^-$ to the norms of $\mathbf{x} \in V$ and $\mathfrak{u}_0 \in S^+$. The mapping $\zeta(\mathfrak{u}_o)$ can be uniquely extended to an involutive automorphism in $V \oplus S^-$. If $\mathfrak{v} \in S^-$ is such that $\mathfrak{v} = \zeta(\mathfrak{u}_0)\mathbf{x}$ for an unique $\mathbf{x} \in V$, then one can define

$$\zeta(\mathfrak{u}_0)(\mathfrak{v}) = \mathbf{x}. \tag{3.2.11}$$

Moreover, it is possible to define $\zeta(\mathfrak{u}_0)$ to act on elements in S^+ as a reflection with respect to the spinor $\mathfrak{u}_0 \in S^+$, namely:

$$\zeta(\mathfrak{u}_0)(\mathfrak{u}) = 2h(\mathfrak{u}, \mathfrak{u}_0)\mathfrak{u}_0 - \mathfrak{u}. \tag{3.2.12}$$

Lemma 3.2. *Let $\mathfrak{u}_0 \in S^+$ and let $\mathbf{x}_0 \in V$ such that $\mathbf{x}_0^2 = 1$ and $h(\mathfrak{u}_0,\mathfrak{u}_0) = 1$. Then, $\zeta(\mathfrak{u}_0)Y(\mathbf{x}_0)\zeta(\mathfrak{u}_0) = Y(\mathbf{x}_0)\zeta(\mathfrak{u}_0)Y(\mathbf{x}_0)$.*

Ultimately, define the operator $\Theta : E \to E$ for elements $\mathbf{x}_0$ and $\mathfrak{u}_0$ as above by

$$\Theta(\mathbf{x}_0, \mathfrak{u}_0) = Y(\mathbf{x}_0)\zeta(\mathfrak{u}_0). \tag{3.2.13}$$

Theorem 3.3. *The operator $\Theta(\mathbf{x}_0, \mathfrak{u}_0)$ is an order 3 automorphism.*

Proof. It is straightforward to see, using Lemma 3.2, that

$$\begin{aligned} \Theta^3(\mathbf{x}_0, \mathfrak{u}_0) &= Y(\mathbf{x}_0)\zeta(\mathfrak{u}_0)Y(\mathbf{x}_0)\zeta(\mathfrak{u}_0)Y(\mathbf{x}_0)\zeta(\mathfrak{u}_0) \\ &= Y(\mathbf{x}_0)\zeta(\mathfrak{u}_0)Y(\mathbf{x}_0)Y(\mathbf{x}_0)\zeta(\mathfrak{u}_0)Y(\mathbf{x}_0) = I. \end{aligned}$$

□

Notice that the operator satisfies

$$\begin{aligned}
\Theta(\mathbf{x}_0, \mathfrak{u}_0)(\mathbf{x}) &= Y(\mathbf{x}_0)\zeta(\mathfrak{u}_0)\mathbf{x} = Y(\mathbf{x}_0)(\mathbf{x}\mathfrak{u}_0) = \mathbf{x}_0\mathbf{x}\mathfrak{u}_0 \in S^+, \\
\Theta(\mathbf{x}_0, \mathfrak{u}_0)(\mathfrak{u}) &= Y(\mathbf{x}_0)\zeta(\mathfrak{u}_0)\mathfrak{u} = Y(\mathbf{x}_0)[2h(\mathfrak{u}, \mathfrak{u}_0)\mathfrak{u}_0 - \mathfrak{u}] \\
&= 2h(\mathfrak{u}, \mathfrak{u}_0)\mathbf{x}\mathfrak{u}_0 - \mathbf{x}_0\mathfrak{u} \in S^-, \\
\Theta(\mathbf{x}_0, \mathfrak{u}_0)(\mathfrak{v}) &= Y(\mathbf{x}_0)\zeta(\mathfrak{u}_0)\mathfrak{v} = Y(\mathbf{x}_0)\mathbf{x} = \mathbf{x}_0\mathbf{x}\mathbf{x}_0 \in V,
\end{aligned}$$

that is,

$$\Theta(\mathbf{x}_0, \mathfrak{u}_0)V \subset S^+, \quad \Theta(\mathbf{x}_0, \mathfrak{u}_0)S^+ \subset S^-, \quad \Theta(\mathbf{x}_0, \mathfrak{u}_0)S^- \subset V. \tag{3.2.14}$$

One says that $\Theta(\mathbf{x}_0, \mathfrak{u}_0)$ cyclically permutates V, S^+ and S^-. It can also be proven [22] that

$$B(\Theta(\mathbf{x}_0, \mathfrak{u}_0)\phi_1, \Theta(\mathbf{x}_0, \mathfrak{u}_0)\phi_2) = B(\phi_1, \phi_2), \tag{3.2.15}$$

$$T(\Theta(\mathbf{x}_0, \mathfrak{u}_0)\phi_1, \Theta(\mathbf{x}_0, \mathfrak{u}_0)\phi_2, \Theta(\mathbf{x}_0, \mathfrak{u}_0)\phi_3) = T(\phi_1, \phi_2, \phi_3). \tag{3.2.16}$$

Therefore, $\Theta(\mathbf{x}_0, \mathfrak{u}_o)$ is a cyclic automorphism that preserves the metric in E and, thus, in each subspace. That way, it is possible to prove [5, 22] that if we take $S^\pm$ as a vector space, then the semispinor space associated with $S^\pm$ shall be $V \oplus S^\mp$. One can further analyze the octonionic algebra $\mathbb{O}$, where geometric realizations can be derived in relation to the triality principle. More details can be found in refs. [5, 22, 24]

4 Conclusion

The Clifford algebras have been defined and analyzed, being the *universal* Clifford algebras for the quadratic space $\mathbb{R}^{p,q}$ the core structures studied throughout this work. Their construction by the quotient of the tensor algebra by a two-sided ideal was implemented, as well as the Atiyah-Bott-Shapiro classification theorem, which shows that every Clifford algebra over the quadratic space $\mathbb{R}^{p,q}$ is isomorphic to a simple algebra $\mathbb{K} \otimes \mathcal{M}(n, \mathbb{R})$, where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Spinors have been effectively studied by encompassing two definitions: the algebraic and classical ones, having each been interpreted and classified. The algebraic spinors were associated to the irreducible regular representation for the Clifford algebras whereas the classical ones emerged from representations on orthogonal transformations in such algebras. Furthermore, the Dirac spinor, which carries the irreducible representations of the left and right handed representations of the Lorentz group double cover $\mathrm{SL}(2, \mathbb{C})$, was examined, its bilinear covariants were discussed and the Fierz identities, along with the reconstruction theorem, was inspected. Then, the Lounesto classification for spinors in Clifford algebra for the Minkowski space $\mathbb{R}^{1,3}$ was presented.

Finally, inner products in the (algebraic) spinor spaces were defined, leading to a construction of the triality principle in the Clifford algebra context. Moreover, this work has enabled us to publish a paper entitled ”Additional fermionic fields onto parallelizable 7-spheres”, where generalized Fierz aggregates were employed to establish a classification for spinors in the 7-sphere S^7 spinor bundle, considering its bilinear covariants and the reconstruction theorem. Then, additional classes of fermionic (spinor) fields on the parallelizable S^7 were constructed, according to the classes previously obtained, lifted from the S^7 spin bundle, with the right transformation under infinitesimal transformations. More details can be seen in [25, 26].

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The Appendix is devoted to compile auxiliary results to the main work, being the parts subject to such complements explicitly indicated throughout the text. The references used in the parts A-G are [5, 6], and the content in part H was proposed as en exercise in ref. [27]. Finally, we give a direct proof of two of the Fierz identities in part I, which are extensively used in refs. [11, 12, 13, 14, 15, 16].

5.1 Appendix A - Preliminaries

We follow to compile some results from linear algebra, which are fundamental for the development of this work.

Tensor Algebra

The tensor product of vector spaces V and W is a natural generalization when considering bilinear maps $\phi : V \times W \to U$. Such product is a vector space T with a bilinear application

$$\begin{aligned} \otimes : V \times W &\to T \\ (\mathbf{v}, \mathbf{w}) &\mapsto \mathbf{v} \otimes \mathbf{w} \end{aligned}$$

satisfying the following condition: if $\{\mathbf{e}_i\}$ is a basis for V and $\{\mathbf{f}_i\}$ is a basis for W then $\{\mathbf{e}_i \otimes \mathbf{f}_i\}$ is a basis for T.

One can, for instance, define the action of the tensor product $\alpha \otimes \beta : V \times V \to \mathbb{R}$ as $(\mathbf{v}, \mathbf{u}) \mapsto \alpha(\mathbf{v})\beta(\mathbf{u})$. Similarly, one could define $\mathbf{v} \otimes \mathbf{u}$ as a bilinear form acting on $V^* \times V^*$. It is clear that the tensor product is not commutative, that is, $\mathbf{v} \otimes \mathbf{u} \neq \mathbf{u} \otimes \mathbf{v}$.

One denotes the tensor product between two copies of V by $T_2(V) = V \otimes V$, which is called a $(0,2)$ contravariant tensor. This notation can be extended for a finite integer q, that is, we can construct a set of $(0,q)$ tensors and denote it by $T_q = (V)^{\otimes q}$. Analogously, one can define $T^p(V) = (V^*)^{\otimes p}$, the set of $(p,0)$ covariant tensors. Moreover, $T_q^p(V) = (V)^{\otimes q} \otimes (V^*)^{\otimes p}$, the set of (p,q) tensors.

A basis for $T_q^p(V)$ is given by the tensor product elements:

$$e^{\mu_1} \otimes e^{\mu_2} \otimes \cdots \otimes e^{\mu_p} \otimes \mathbf{e}_{\nu_1} \otimes \mathbf{e}_{\nu_2} \otimes \cdots \otimes \mathbf{e}_{\nu_q}, \tag{5.1.1}$$

and a generic element $T \in T_q^p(V)$ can be written as:

$$T = T^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p} e^{\mu_1} \otimes e^{\mu_2} \otimes \cdots \otimes e^{\mu_p} \otimes \mathbf{e}_{\nu_1} \otimes \mathbf{e}_{\nu_2} \otimes \cdots \otimes \mathbf{e}_{\nu_q}, \tag{5.1.2}$$

where $\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q = 1, \ldots, n$ and

$$T^{\nu_1\nu_2\ldots\nu_q}_{\mu_1\mu_2\cdots\mu_p} = T(\mathbf{e}_{\mu_1}, \ldots, \mathbf{e}_{\mu_p}, e^{\nu_1}, \ldots, e^{\nu_q}). \tag{5.1.3}$$

Let T and S be two (p,q) tensors. Their sum can be defined as the (p,q) tensor $(T+S)$ as follows:

$$(T+S)^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p} = T^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p} + S^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p}. \tag{5.1.4}$$

Now, if T is a (p,q) tensor and S is a (r,s) tensor, one can define the tensor product $T \otimes S$, which is a $(p+r, q+s)$ tensor. Component-wise:

$$(T \otimes S)^{\nu_1\nu_2\cdots\nu_q\rho_1\cdots\rho_s}_{\mu_1\mu_2\cdots\mu_p\sigma_1\cdots\sigma_r} = T^{\nu_1\nu_2\cdots\nu_q}_{\mu_1\mu_2\cdots\mu_p} S^{\rho_1\rho_2\cdots\rho_r}_{\sigma_1\sigma_2\cdots\sigma_s}. \tag{5.1.5}$$

The product $\otimes$ is distributive and associative regarding the sum and therefore, there is an algebra structure in the direct sum of all $T_q^p(V)$ spaces equipped with both the above operations, called the *tensor algebra*. The contravariant tensor algebra is the tensor algebra of $(0,q)$ tensors, denoted by $T(V) = \oplus_{q=0}^{\infty} T_q(V)$, whereas $T^*(V) = \oplus_{p=0}^{\infty} T^p(V)$ is the covariant tensor algebra of $(p,0)$ tensors.

One can define applications over the elements of the tensor algebra: let $T, S \in T^p(V)$ be $(p,0)$ tensors (the same can be done to $(0,q)$ tensors) and define $\widehat{\ }$ (or #, depending on which symbol is easier to use) as the *grade involution* and $\sim$ the *reversion* by the relations:

$$\#T = \widehat{(T)} = (-1)^p T \quad \text{and} \quad \widetilde{T \otimes S} = \widetilde{S} \otimes \widetilde{T}. \tag{5.1.6}$$

Then, it follows that

$$(\alpha \otimes \widetilde{\beta \otimes \cdots} \otimes \omega) = \omega \otimes \cdots \otimes \beta \otimes \alpha. \tag{5.1.7}$$

Finally, the composition between the grade involution and reversion is called *conjugation* and defined as

$$\overline{T} = \widehat{\widetilde{T}} = \widetilde{\widehat{T}}. \tag{5.1.8}$$

Let now $X_1 \otimes X_2 \otimes \cdots \otimes X_p$ be a p-tensor and σ a permutation of the set $\{1, 2, \ldots, p\}$. The alternating operator ALT can be defined as it follows:

$$ALT(X_1 \otimes X_2 \otimes \cdots X_p) = \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon(\sigma) X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \cdots X_{\sigma(p-1)} \otimes X_{\sigma(p)}, \tag{5.1.9}$$

where S_p is the set of permutations of $\{1, \ldots, p\}$ and $\varepsilon(\sigma)$ is the sign of the permutation σ, that is, $\varepsilon(\sigma) = +1(-1)$ if σ is an even (odd) permutation.

Exterior Algebra

One says that a p-vector is a contravariant alternating tensor of order p, denoted $A_{[p]}$. Analogously, $\psi^{[p]}$ is a covariant alternating tensor of order p, where alternating means that $ALT(A_{[p]}) = A_{[p]}$. The p-vector space is denoted $\Lambda_p(V)$ whereas the p-covector space is $\Lambda^p(V)$.

It is clear that $\dim \Lambda_p(V) = \dim \Lambda^p(V)$. A simple combinatory calculation can be computed to find these spaces dimensions: let $B^p = \{e^{i_1}, e^{i_2}, \ldots, e^{i_p}\}$ be a basis for $T^p(V)$, where $i_k = 1, 2, \ldots, n; (k = 1, 2, \ldots, p)$. Since the ALT application cancels the terms with repeating labels, for i_1 we have n choices, for i_2 there are $(n-1)$ choices, and so on until i_p where there are $(n-p+1)$ choices. Moreover, each permutation of labels will give the same result after alternation. Therefore, $\dim \Lambda^p(V) = \frac{n(n-1)\cdots(n-p+1)}{p!} = \frac{n!}{(n-p)!p!} = \binom{n}{p}$

Now, $\binom{n}{p} = \binom{n}{n-p}$ yields

$$\dim \Lambda^p(V) = \dim \Lambda^{n-p}(V). \tag{5.1.10}$$

Let a $A_{[p]}$ be a p-vector and $B_{[q]}$ a q-vector. Taking the tensor product, the result is indeed a contravariant tensor of order $p+q$ but it is not necessarily alternating. We can then take $ALT(A_{[p]} \otimes B_{[q]})$, which is an alternating contravariant tensor of order $(p+q)$.

Definition 5.1. *Let $A_{[p]} \in \Lambda_p(V)$ a p-vector and $B_{[q]} \in \Lambda_q(V)$ a q-vector. The exterior product $\wedge : \Lambda_p(V) \times \Lambda_q(V) \to \Lambda_{p+q}(V)$ is defined as:*

$$A_{[p]} \wedge B_{[q]} = ALT(A_{[p]} \otimes B_{[q]}). \tag{5.1.11}$$

Naturally, the exterior product inherits from the tensor product associativity and bilinearity. By definition, for two vectors $\mathbf{v}, \mathbf{u} \in V$ one has

$$\mathbf{v} \wedge \mathbf{u} = \frac{1}{2}(\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}), \tag{5.1.12}$$

which implies $\mathbf{v} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{v}$ and $\mathbf{v} \wedge \mathbf{v} = 0$, that is, the exterior product is *anti-commutative*.

Let $A_{[p]}$ and $B_{[q]}$ be a p-vector and a q-vector, respectively, such that

$$A_{[p]} = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_p, \; B_{[q]} = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_q. \tag{5.1.13}$$

Then, the exterior product $A_{[p]} \wedge B_{[q]}$ can be written as

$$A_{[p]} \wedge B_{[q]} = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_p \wedge \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_q, \tag{5.1.14}$$

and using that $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$, one gets the following result:

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_p \wedge \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_q$$

$$= (-1)^{pq} \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_q \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_p,$$

that is

$$A_{[p]} \wedge B_{[q]} = (-1)^{pq} B_{[q]} \wedge A_{[p]}. \tag{5.1.15}$$

Consider now the vector space $\Lambda(V) = \oplus_{p=0}^{n} \Lambda_p(V)$. It is clear that $\wedge : \Lambda(V) \times \Lambda(V) \rightarrow \Lambda(V)$, that is, the exterior product (which is bilinear) is closed over $\Lambda(V)$. Hence, the pair $(\Lambda(V), \wedge)$ is an algebra, denoted the *exterior algebra* of V. An arbitrary element of $\Lambda(V)$ is called a *multivector*. It is a sum of a scalar, a vector, a 2-vector and so on, up to the n-vector. Namely, we have:

$$\Lambda(V) \ni A = a + v^i \mathbf{e}_i + F^{ij} \mathbf{e}_i \wedge \mathbf{e}_j + T^{ijk} \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k + \cdots + p \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n \tag{5.1.16}$$

The dimension of $\Lambda(V)$ is given by

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{p=0}^{n} \binom{n}{p} = (1+1)^n = 2^n \tag{5.1.17}$$

We denote $\langle \cdot \rangle_p$ the *projector* $\langle \cdot \rangle_p : \Lambda(V) \rightarrow \Lambda_p(V)$. If $A \in \Lambda(V)$, then $\langle A \rangle_p = A_{[p]} \in \Lambda_p(V)$, the p-vector part of A. Also, the operations defined for the tensor algebra are naturally inherited for the exterior algebra. For the grade involution, we have

$$\# A_{[p]} = \widehat{A_{[p]}} = (-1)^p A_{[p]}, \tag{5.1.18}$$

whereas for the reversion

$$\widetilde{(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p)} = \mathbf{v}_p \wedge \cdots \wedge \mathbf{v}_1, \tag{5.1.19}$$

which implies that

$$\widetilde{A}_{[p]} = (-1)^{\frac{p(p-1)}{2}} A_{[p]}. \tag{5.1.20}$$

Finally, the conjugation, which is the composition of the previous ones, is still given by

$$\overline{A}_{[p]} = \widetilde{\widehat{A_{[p]}}} = \widehat{\widetilde{A_{[p]}}}. \tag{5.1.21}$$

Exterior Algebra as a Quotient of Tensor Algebra

An equivalence relation R in a set X is a *reflexive (1)*, *symmetric (2)* and *transitive (3)* relation, that is:

1. $xRx,\ \forall x$
2. $xRy \Rightarrow yRx$
3. xRy and $yRz \Rightarrow xRz$

The set of all elements which are equivalent to x is the *equivalence class* of x, denoted $[x] = \{y \in X : yRx\}$. The set of these equivalence classes (which are one-to-one disjoint) is denoted $\mathcal{X} = X/R = \{[x] : x \in X\}$. Also, one can write $x \sim y$ to mean xRy.

Let $\mathcal{A}$ be an algebra. A set $J \subset A$ is said to be an *ideal* of $\mathcal{A}$ if for every $a, b \in \mathcal{A}$ and $x \in J$ it follows that $axb \in J$. Now, write $\mathcal{A} = \mathcal{B} + \mathcal{C}$ and define in $\mathcal{A}$ the following equivalence relation:

$$a \sim b \iff a = b + x,\ \ x \in \mathcal{C}. \tag{5.1.22}$$

The set $\mathcal{A}/\sim$ of equivalence classes has a natural vector space structure. One can define the sum of vectors as

$$[a] + [b] = [a + b], \tag{5.1.23}$$

and the scalar multiplication

$$\alpha[a] = [\alpha a]. \tag{5.1.24}$$

We investigate if $\mathcal{A}/\sim$ as constructed is an algebra. The natural way to define the product of two elements of $\mathcal{A}/\sim$ is by

$$[a][b] = [ab]. \tag{5.1.25}$$

By the definition, we know that $[a] = [a+x]$ and $[b] = [b+y]$ where $x, y \in \mathcal{C}$. It follows that

$$[a][b] = [a+x][b+y] = [(a+x)(b+y)] = [ab+ay+xb+xy]. \tag{5.1.26}$$

Therefore, in order to $\mathcal{A}/\sim$ be an algebra, we would need that $ay+xb+xy \in \mathcal{C}$, that is, $\mathcal{C}$ must be an ideal of $\mathcal{A}$. In that case, we call $\mathcal{A}/\sim$ the quotient algebra of $\mathcal{A}$, and denote it as $\mathcal{A}/\mathcal{C}$.

Let $T(V)$ be the algebra of the contravariant tensors. Consider the ideal I of $T(V)$ generated by elements such as $\mathbf{v}\otimes\mathbf{v}$ for each $\mathbf{v}\in V$. Then, elements in I have the general form of

$$\sum_i A_i \otimes \mathbf{v}_i \otimes \mathbf{v}_i \otimes B_i, \tag{5.1.27}$$

where $\mathbf{v}_i \in V$ and $A_i, B_i \in T(V)$. One can see that the ideal I is generated by elements such as $\mathbf{v}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{v}$, since

$$\mathbf{v}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{v} = (\mathbf{v}+\mathbf{u})\otimes(\mathbf{v}+\mathbf{u}) - \mathbf{v}\otimes\mathbf{v} - \mathbf{u}\otimes\mathbf{u}. \tag{5.1.28}$$

Consider now $T(V)/I$ with the equivalence relation

$$A \sim B \iff A = B + x,\ x \in I. \tag{5.1.29}$$

Define the product

$$[A]\wedge[B] = [A\otimes B] \tag{5.1.30}$$

and let $\mathbf{v}, \mathbf{u} \in V$. We can notice that

$$\mathbf{v}\otimes\mathbf{u} = \frac{1}{2}(\mathbf{v}\otimes\mathbf{u}-\mathbf{u}\otimes\mathbf{v}) + \frac{1}{2}(\mathbf{v}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{v}), \tag{5.1.31}$$

where $\frac{1}{2}(\mathbf{v}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{v}) \in I$. Thus we have

$$[\mathbf{v}\otimes\mathbf{u}] = [\frac{1}{2}(\mathbf{v}\otimes\mathbf{u}-\mathbf{u}\otimes\mathbf{v})] = [ALT(\mathbf{v}\otimes\mathbf{u})], \tag{5.1.32}$$

which can be generalized by

$$\mathbf{v}_1\otimes\cdots\otimes\mathbf{v}_p \sim ALT(\mathbf{v}_1\otimes\cdots\otimes\mathbf{v}_p) = \mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_p, \tag{5.1.33}$$

establishing then the isomorphism

$$\Lambda(V) \simeq T(V)/I. \tag{5.1.34}$$

Consider the linear application $\alpha : V \to \mathbb{R}$. In the exterior algebra notation, α is an application from $\Lambda_1(V) = V$ to $\Lambda_0(V) = \mathbb{R}$. Generalizing the idea, let $A_{[p]}$ be a p-vector and α a covector. We desire to define an operation acting over $A_{[p]} \in \Lambda_p(V)$ whose value will be in $\Lambda_{p-1}(V)$. We define the left contraction by the covector α, denoted as $\alpha\rfloor$, by

$$(\alpha\rfloor A_{[p]})(\alpha_1,\alpha_2,\ldots,\alpha_{p-1}) = pA_{[p]}(\alpha,\alpha_1,\ldots,\alpha_{p-1}), \tag{5.1.35}$$

where $\alpha_1,\ldots,\alpha_{p-1}$ are arbitrary covectors and such that if $A_{[p]} = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_p$, then

$$(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p)(\alpha,\alpha_1,\ldots,\alpha_{p-1}) = \frac{1}{p!}\sum_{\sigma\in S_p}\varepsilon(\sigma)\alpha(\mathbf{v}_{\sigma(1)})\alpha_1(\mathbf{v}_{\sigma_2})\cdots\alpha_{p-1}(\mathbf{v}_{\sigma(p)}). \tag{5.1.36}$$

Hence, the definition above makes it clear that $\alpha\rfloor A_{[p]}$ is a $(p-1)$-vector.Taking a vector $\mathbf{v}$ the expression is simply

$$\alpha\rfloor\mathbf{v} = \alpha(\mathbf{v}), \tag{5.1.37}$$

whereas for an element $a \in \Lambda_0(V)$ we assume the value is zero, thus $\alpha\rfloor a = 0$. Let us consider a left contraction of a 2-vector $\mathbf{v}\wedge\mathbf{u}$. Using the definitions, we have that

$$(\alpha\rfloor(\mathbf{v}\wedge\mathbf{u}))(\beta) = 2(\mathbf{v}\wedge\mathbf{u})(\alpha,\beta) = \alpha(\mathbf{v})\beta(\mathbf{u}) - \alpha(\mathbf{u})\beta(\mathbf{v})$$

$$= (\alpha(\mathbf{v})\mathbf{u} - \alpha(\mathbf{u})\mathbf{v})\beta = ((\alpha\rfloor\mathbf{v})\mathbf{u} - (\alpha\rfloor\mathbf{u})\mathbf{v})(\beta).$$

Since the expression is valid for any covector β, it follows that

$$\alpha\rfloor(\mathbf{v}\wedge\mathbf{u}) = (\alpha\rfloor\mathbf{v})\mathbf{u} - \mathbf{v}(\alpha\rfloor\mathbf{u}) = \alpha(\mathbf{v})\mathbf{u} - \mathbf{v}\alpha(\mathbf{u}). \tag{5.1.38}$$

Now, for a 3-vector $\mathbf{v}\wedge\mathbf{u}\wedge\mathbf{w}$ we can see that

$$(\alpha\rfloor(\mathbf{v}\wedge\mathbf{u}\wedge\mathbf{w}))(\beta,\gamma) = 3(\mathbf{v}\wedge\mathbf{u}\wedge\mathbf{w})(\alpha,\beta,\gamma)$$

$$= \frac{3}{3!}[\alpha(\mathbf{v})\beta(\mathbf{u})\gamma(\mathbf{w}) + \beta(\mathbf{v})\gamma(\mathbf{u})\alpha(\mathbf{w}) + \gamma(\mathbf{v})\alpha(\mathbf{u})\beta(\mathbf{w})$$
$$-\gamma(\mathbf{v})\beta(\mathbf{u})\alpha(\mathbf{w}) - \beta(\mathbf{v})\alpha(\mathbf{u})\gamma(\mathbf{w}) - \alpha(\mathbf{v})\gamma(\mathbf{u})\beta(\mathbf{w})]$$
$$= \alpha(\mathbf{v})(\frac{1}{2!})[\beta(\mathbf{u})\gamma(\mathbf{w}) - \beta(\mathbf{w})\gamma(\mathbf{u})]$$

$$-\alpha(\mathbf{u})(\frac{1}{2!})[\beta(\mathbf{v})\gamma(\mathbf{w})-\beta(\mathbf{w})\gamma(\mathbf{v})]$$
$$+\alpha(\mathbf{w})(\frac{1}{2!})[\beta(\mathbf{v})\gamma(\mathbf{u})-\beta(\mathbf{u})\gamma(\mathbf{v})]$$
$$=\alpha(\mathbf{v})(\mathbf{u}\wedge\mathbf{w})(\beta,\gamma)-\alpha(\mathbf{u})(\mathbf{v}\wedge\mathbf{w})(\beta,\gamma)+\alpha(\mathbf{w})(\mathbf{v}\wedge\mathbf{u})(\beta,\gamma),$$

and since it must be valid for every β and γ, there holds

$$\begin{aligned}\alpha\rfloor(\mathbf{v}\wedge\mathbf{u}\wedge\mathbf{w}) &= (\alpha\rfloor\mathbf{v})\mathbf{u}\wedge\mathbf{w}-(\alpha\rfloor\mathbf{u})\mathbf{v}\wedge\mathbf{w}+(\alpha\rfloor\mathbf{w})v\wedge\mathbf{u}\\ &= \alpha(\mathbf{v})\mathbf{u}\wedge\mathbf{w}-\alpha(\mathbf{u})\mathbf{v}\wedge\mathbf{w}+\alpha(\mathbf{w})\mathbf{v}\wedge\mathbf{u}.\end{aligned} \tag{5.1.39}$$

Finally, using the equation for the 2-vector case and the exterior product associativity, we conclude that:

$$\alpha\rfloor((\mathbf{v}\wedge\mathbf{u})\wedge\mathbf{w})=(\alpha\rfloor(\mathbf{v}\wedge\mathbf{u}))\wedge\mathbf{w}+\mathbf{v}\wedge(\alpha\rfloor\mathbf{w}), \tag{5.1.40}$$

which is the contraction of the exterior product of a 2-vector and a 1-vector, whereas

$$\alpha\rfloor(\mathbf{v}\wedge(\mathbf{u}\wedge\mathbf{w}))=(\alpha\rfloor\mathbf{v})\mathbf{u}\wedge\mathbf{w}-\mathbf{v}\wedge(\alpha\rfloor(\mathbf{u}\wedge\mathbf{w})) \tag{5.1.41}$$

is a contraction of the exterior product of a 1-vector and a 2-vector. The generalization of these equations for the exterior product of a q-vector and p-vector can be written as

$$\alpha\rfloor(A_{[p]}\wedge B_{[q]})=(\alpha\rfloor A_{[p]})\wedge B_{[q]}+(-1)^pA_{[p]}\wedge(\alpha B_{[q]}). \tag{5.1.42}$$

Such equation resembles the Leibniz rule for derivation and we will call it the *grade* Leibniz rule, due to the grade involution presence. We can also define the right contraction by α, denoted $\lfloor\alpha$, by

$$(A[p]\lfloor\alpha)(\alpha_1,\alpha_2,\ldots,\alpha_{p-1})=pA_{[p]}(\alpha_1,\alpha_2,\ldots,\alpha_{p-1},\alpha), \tag{5.1.43}$$

where α_i are arbitrary covectors. For a scalar $a\in\mathbb{R}$ and a vector $\mathbf{v}\in\mathbb{R}^3$ we have the same results, naturally:

$$\mathbf{v}\lfloor\alpha=\alpha(\mathbf{v}),\quad a\lfloor\alpha=0. \tag{5.1.44}$$

The difference between the right and left contractions begin when we analyse the case for a 2-vector. Using the same arguments for the left contraction, we get that

$$(\mathbf{v}\wedge\mathbf{u})\lfloor\alpha=\mathbf{v}(\mathbf{u}\lfloor\alpha)-\mathbf{u}(\mathbf{v}\lfloor\alpha)=\alpha(\mathbf{u})\mathbf{v}-\alpha(\mathbf{v})\mathbf{u}. \tag{5.1.45}$$

Using similar arguments, we can formulate the analogous general equation for the right contraction:

$$(A_{[p]} \wedge B_{[q]}) \lfloor \alpha = A_{[p]} \wedge (B_{[q]} \lfloor \alpha) + (A_{[p]} \lfloor \alpha) \wedge (-1)^q B_{[q]}. \tag{5.1.46}$$

Finally, comparing the equations, we have that

$$A_{[p]}(\alpha_1, \ldots, \alpha_{p-1}, \alpha) = (-1)^{p-1} A_{[p]}(\alpha, \alpha_1, \ldots, \alpha_{p-1}), \tag{5.1.47}$$

which implies that

$$\alpha \lfloor A_{[p]} = (-1)^{p-1} A_{[p]} \rfloor \alpha. \tag{5.1.48}$$

5.2 Appendix B - $T(V)/I$ as the Universal Clifford Algebra

We follow to construct $T(V)/I$ using the equivalence relation

$$A \sim B \iff A = B + x, \ x \in I, \tag{5.2.1}$$

denoting the product of the equivalence classes by juxtaposition, that is:

$$[A][B] = [A \otimes B]. \tag{5.2.2}$$

It is known that the elements in the ideal I are written as

$$\sum_i A_i \otimes (\mathbf{v} \otimes \mathbf{v} - Q(v)1) \otimes B_i, \tag{5.2.3}$$

where $A_i, B_i \in T(V)$. We could also say that the generating elements of I are

$$\mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} - 2g(\mathbf{v}, \mathbf{u}), \tag{5.2.4}$$

since $(\mathbf{u} + \mathbf{v}) \otimes (\mathbf{u} + \mathbf{v}) - g(\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v})$ is a generating element of the ideal and

$$(\mathbf{u} + \mathbf{v}) \otimes (\mathbf{u} + \mathbf{v}) - 2g(\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}) \sim \mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} - 2g(\mathbf{v}, \mathbf{u}). \tag{5.2.5}$$

Then, let $\mathbf{v}, \mathbf{u} \in V$ and consider the tensor product $\mathbf{v} \otimes \mathbf{u}$. One can write such product as

$$\begin{aligned}\mathbf{v} \otimes \mathbf{u} =& \frac{1}{2}(\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}) + g(\mathbf{v}, \mathbf{u}) + \frac{1}{2}((\mathbf{v} + \mathbf{u}) \otimes (\mathbf{v} + \mathbf{u}) \\ &- g(\mathbf{v} + \mathbf{u}, \mathbf{v} + \mathbf{u}) - \mathbf{v} \otimes \mathbf{v} + g(\mathbf{v}, \mathbf{v}) - \mathbf{u} \otimes \mathbf{u} + g(\mathbf{u}, \mathbf{u})),\end{aligned} \tag{5.2.6}$$

where the last term is in I. Therefore

$$\mathbf{v} \otimes \mathbf{u} \sim \frac{1}{2}(\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}) + g(\mathbf{v}, \mathbf{u}), \tag{5.2.7}$$

or even in terms of exterior product:

$$\mathbf{v} \otimes \mathbf{u} \sim \mathbf{v} \wedge \mathbf{u} + g(\mathbf{v}, \mathbf{u}). \tag{5.2.8}$$

Hence, removing the bracket notation for the product in the quotient, it follows that

$$\mathbf{v}\mathbf{u} = [\mathbf{v} \wedge \mathbf{u} + g(\mathbf{v}, \mathbf{u})], \tag{5.2.9}$$

and since the sum of classes is defined as the class of the sum, and since $[g(\mathbf{u}, \mathbf{v})] = g(\mathbf{u}, \mathbf{v})[1]$, which will be identified as $g(\mathbf{u}, \mathbf{v})$. Then,

$$\mathbf{v}\mathbf{u} = [\mathbf{v} \wedge \mathbf{u}] + g(\mathbf{v}, \mathbf{u}). \tag{5.2.10}$$

Now, we could loose the bracket notation for the exterior product if the quotient application $[\,\cdot\,]$ was injective in $\Lambda(V)$. Indeed, that is the case. Notice that $[A] = 0$ if, and only if $A \in I$. Also, suppose A is non null. Then, there are $A_i, B_i \in T(V)$ such that $A = \sum_i A_i \otimes (\mathbf{v} \otimes \mathbf{v} - Q(\mathbf{v})1) \otimes B_i$. Notice that $ALT(\sum_i A_i \otimes (\mathbf{v} \otimes \mathbf{v}) \otimes B_i) = 0$, since that for every permutation in the sum there is another that changes the sign and interchanges $\mathbf{v}$ with itself. Therefore

$$\begin{aligned}&ALT\Big[\sum_i A_i \otimes (\mathbf{v} \otimes \mathbf{v} - Q(\mathbf{v})1) \otimes B_i\Big] \\ =&ALT\Big[\sum_i A_i \otimes \mathbf{v} \otimes \mathbf{v} \otimes B_i\Big] - ALT\Big[\sum_i A_i \otimes Q(\mathbf{v})1 \otimes B_i\Big] \\ =& -Q(\mathbf{v})ALT\Big[\sum_i A_i \otimes 1 \otimes B_i\Big],\end{aligned} \tag{5.2.11}$$

which is not A, provided A is non null, since the last term above lives in another tensor dimension in $T(V)$ than A. Therefore, $ALT(A) \neq A$ and thus $[\,\cdot\,]$ is injective in

$\Lambda(V)$. It follows that

$$\mathbf{vu} = \mathbf{v} \wedge \mathbf{u} + g(\mathbf{v}, \mathbf{u}). \tag{5.2.12}$$

It can be shown that a similar equation holds for the case of product between bivectors and vectors: let $\mathbf{v}, \mathbf{u}, \mathbf{w} \in V$. The exterior product $\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w}$ can be written as it follows:

$$\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w} = \frac{1}{3}(\mathbf{v} \otimes (\mathbf{u} \wedge \mathbf{v}) - \mathbf{u} \otimes (\mathbf{v} \wedge \mathbf{w}) + \mathbf{w} \otimes (\mathbf{v} \wedge \mathbf{u})). \tag{5.2.13}$$

Tensoring both sides of Eq.(5.2.8) yields the relations

$$\begin{aligned} \mathbf{u} \otimes (\mathbf{v} \wedge \mathbf{w}) &\sim \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} - g(\mathbf{v}, \mathbf{w})\mathbf{u}, \\ \mathbf{w} \otimes (\mathbf{v} \wedge \mathbf{u}) &\sim \mathbf{w} \otimes \mathbf{v} \otimes \mathbf{u} - g(\mathbf{v}, \mathbf{u})\mathbf{w}. \end{aligned} \tag{5.2.14}$$

Since $\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u} - 2g(\mathbf{u}, \mathbf{v}) \sim 0$, it holds that

$$\begin{aligned} \mathbf{u} \otimes \mathbf{v} &= 2g(\mathbf{u}, \mathbf{v}) - \mathbf{v} \otimes \mathbf{u}, \\ \mathbf{w} \otimes \mathbf{v} &\sim 2g(\mathbf{w}, \mathbf{v}) - \mathbf{v} \otimes \mathbf{w}, \end{aligned} \tag{5.2.15}$$

which implies that

$$\begin{aligned} \mathbf{u} \otimes (\mathbf{v} \wedge \mathbf{w}) &\sim -\mathbf{v} \otimes \mathbf{u} \otimes \mathbf{w} + 2g(\mathbf{u}, \mathbf{v})\mathbf{w} - g(\mathbf{v}, \mathbf{w})\mathbf{u}, \\ \mathbf{w} \otimes (\mathbf{v} \wedge \mathbf{u}) &\sim -\mathbf{v} \otimes \mathbf{w} \otimes \mathbf{u} + 2g(\mathbf{w}, \mathbf{v})\mathbf{u} - g(\mathbf{v}, \mathbf{u})\mathbf{w}. \end{aligned} \tag{5.2.16}$$

Now, using the above equalities in Eq.(5.2.13), one gets the result

$$\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w} = \mathbf{v} \otimes (\mathbf{u} \wedge \mathbf{w}) + g(\mathbf{v}, \mathbf{w})\mathbf{u} - g(\mathbf{v}, \mathbf{u})\mathbf{w}. \tag{5.2.17}$$

Recalling the result for the left contraction from Eq.(5.1.38), and using $\mathbf{v}_\flat(\mathbf{u}) = g(\mathbf{v}, \mathbf{u})$, we have that

$$\mathbf{v} \otimes (\mathbf{u} \wedge \mathbf{w}) \sim \mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w} + \mathbf{v}_\flat \rfloor (\mathbf{u} \wedge \mathbf{w}). \tag{5.2.18}$$

Therefore, the product of a 2-vector and a vector in $T(V)/I$ can be written as

$$\mathbf{v}(\mathbf{u} \wedge \mathbf{w}) = \mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w} + \mathbf{v}_\flat \rfloor (\mathbf{u} \wedge \mathbf{w}), \tag{5.2.19}$$

and the equation can be generalized as it follows

$$\mathbf{v}A_{[p]} = \mathbf{v} \wedge A_{[p]} + \mathbf{v}_\flat \rfloor A_{[p]}. \tag{5.2.20}$$

The quotient algebra $T(V)/I$ is a Clifford Algebra, as one can see from Eq.(5.2.12)

$$\mathbf{vu} + \mathbf{uv} = 2g(\mathbf{v}, \mathbf{u}). \tag{5.2.21}$$

The Clifford application is $\gamma = \pi \circ i$ where i is the inclusion $i : V \to T(V)$ and $\pi : T(V) \to \mathcal{C}\ell(V, g) = T(V)/I$, that is, $\gamma(v) = [v]$. We can loose both the bracket and the Clifford application notation and leave it implicit. One can demonstrate an analogous expression to Eq.(5.2.20), which is

$$A_{[p]}\mathbf{v} = A_{[p]} \wedge \mathbf{v} + A_{[p]} \llcorner \mathbf{v}_\flat. \tag{5.2.22}$$

Now, using Eq.(5.1.15) for the exterior product and that $A_{[p]} \llcorner \mathbf{v}_\flat = -(-1)^p \mathbf{v}_\flat \lrcorner A_{[p]}$, one can write that

$$A_{[p]}\mathbf{v} = (-1)^p \mathbf{v} \wedge A_{[p]} - (-1)^p \mathbf{v}_\flat \lrcorner A_{[p]}. \tag{5.2.23}$$

Comparing Eq.(5.6.6) with Eq.(5.2.23), it follows that

$$\mathbf{v} \wedge A_{[p]} = \frac{1}{2}(\mathbf{v}A_{[p]} + (-1)^p A_{[p]}\mathbf{v}) \tag{5.2.24}$$

and

$$\mathbf{v}_\flat \lrcorner A_{[p]} = \frac{1}{2}(\mathbf{v}A_{[p]} - (-1)^p A_{[p]})\mathbf{v}. \tag{5.2.25}$$

5.3 Appendix C - Clifford Algebras Complexification Theorem

Theorem. *Let (V, g) be a quadratic space over $\mathbb{R}$ and $\mathcal{C}\ell(V, g)$ its real Clifford algebra. Consider $\mathcal{C}\ell(V_\mathbb{C}, g_\mathbb{C})$ the complex Clifford algebra for the quadratic complexified space $(V_\mathbb{C}, g_\mathbb{C})$. Then*

$$\mathcal{C}\ell(V_\mathbb{C}, g_\mathbb{C}) \simeq \mathcal{C}\ell_\mathbb{C}(V, g)$$

where $\mathcal{C}\ell_\mathbb{C}(V, g) = \mathbb{C} \otimes \mathcal{C}\ell(V, g)$ is the complexification of $\mathcal{C}\ell(V, g)$.

Proof. The complexification $\mathcal{C}\ell_\mathbb{C}(V, g) = \mathbb{C} \otimes \mathcal{C}\ell(V, g)$ is an algebra with product given by

$$(a \otimes A)(b \otimes B) = ab \otimes AB, \ \ \forall a, b \in \mathbb{C}, \ \ \forall A, B \in \mathcal{C}\ell(V, g). \tag{5.3.1}$$

Since $dim_\mathbb{R}\, \mathcal{C}\ell(V, g) = 2^{\dim V}$, we have that $dim_\mathbb{R}\, \mathcal{C}\ell_\mathbb{C}(V, g) = 2 * 2^{\dim V}$. If γ is the Clifford application $\gamma : V \to \mathcal{C}\ell(V, g)$, an application $\Gamma : V_\mathbb{C} \to \mathcal{C}\ell_\mathbb{C}(V, g)$ can be defined as a linear application in $\mathbb{C}$ by

$$\Gamma = 1 \otimes \gamma, \tag{5.3.2}$$

where 1 is the identity. Therefore, for $a \otimes \mathbf{v} \in \mathbb{C} \otimes V = V_{\mathbb{C}}$ it follows that

$$\Gamma(a \otimes \mathbf{v}) = a \otimes \gamma(\mathbf{v}). \tag{5.3.3}$$

That way, Γ is a Clifford application. Indeed:

$$(\Gamma(1 \otimes \mathbf{v}))^2 = (1 \otimes \gamma(\mathbf{v}))(1 \otimes \gamma(\mathbf{v})) = 1 \otimes g(\mathbf{v}, \mathbf{v}). \tag{5.3.4}$$

Now, since $\mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}))$ is the universal Clifford algebra for $(V_{\mathbb{C}}, g_{\mathbb{C}})$, there is a homomorphism $\varphi : \mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) \to \mathcal{C}\ell_{\mathbb{C}}(V, g)$. The dimension of $\mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}})$ is $dim_{\mathbb{C}}\ (V_{\mathbb{C}}, g_{\mathbb{C}}) = 2^{\dim V}$ or $dim_{\mathbb{R}}\ \mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) = 2 * 2^{\dim V}$. Since $\dim \mathcal{C}\ell(V_{\mathbb{C}}, g_{\mathbb{C}}) = \dim \mathcal{C}\ell_{\mathbb{C}}(V, g)$, we have that φ is, in fact, an isomorphism. □

5.4 Appendix D - Components of O(p, q)

Let G be a group. A path in G is a continuous application $\phi : [0, 1] \to G$. A subset G' of G is connected if for every $g_0, g_1 \in G'$, there is a path $\phi(t)$ such that $\phi(0) = g_0$ and $\phi(1) = g_1$[1]. A connected subset which is not contained in any other bigger connected subset is called a component of G.

The orthogonal groups $\mathrm{O}(n, 0)$ and $\mathrm{O}(0, n)$ have two components. Indeed, let us take $T_0, T_1 \in \mathrm{O}(n, 0)$ such that $\det(T_0) = 1$ and $\det(T_1) = -1$. There is no continuous application between these transformations in $\mathrm{O}(p, q)$. One of the components of $\mathrm{O}(n, 0)$ is $\mathrm{SO}(n, 0)$ and the same applies to $\mathrm{O}(0, n)$.

On the other hand, the orthogonal groups $\mathrm{O}(p, q)$, with $p \neq 0$ and $q \neq 0$, have four components. Consider $\{\mathbf{e}_1, \ldots, \mathbf{e}_p, \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q}\}$ an orthonormal basis for $\mathbb{R}^{p,q}$. That way, one can write

$$G = \begin{pmatrix} 1_p & \mathcal{O} \\ \mathcal{O} & -1_q \end{pmatrix} \text{ and } T = \begin{pmatrix} A_p & B_{p,q} \\ C_{q,p} & D_q \end{pmatrix}, \tag{5.4.1}$$

where 1_n is the $n \times n$ identity matrix and $\mathcal{O}$ is a fitting null matrix. Since $T^T G T = G$,

[1] Here connected in fact means *path-connected*. Not every path-connected space is connected, but the reverse holds and, therefore, path-connectedness is more than enough to show that $\mathrm{O}(p, q)$ is not connected.

it follows that

$$\begin{aligned} A_p^T A_p - C_{p,q}^T C_{q,p} &= 1_p, \\ D_q^T D_q - B_{q,p}^T B_{p,q} &= 1_q, \\ A_p^T B_{p,q} &= C_{p,q}^T D_q. \end{aligned} \tag{5.4.2}$$

Notice that $\det(A_p) \neq 0$ and $\det(D_q) \neq 0$. Indeed, one can see that since

$${}_p^T A_p = 1_p + C_{p,q}^T C_{q,p} \tag{5.4.3}$$

then

$$(\det(A_p))^2 = \det(1_p + C_{p,q}^T C_{q,p}), \tag{5.4.4}$$

and supposing that $\det(1_p + C_{p,q}^T C_{q,p}) = 0$, it follows that $(1_p + C_{p,q}^T C_{q,p})X = \mathcal{O}_p$ has a non-trivial solution. Namely, $X = -C_{p,q}^T C_{q,p} X$. It follows that

$$X^T X = -X^T C_{p,q}^T C_{q,p} X = -(C_{q,p} X)^T (C_{q,p} X), \tag{5.4.5}$$

that is

$$(\det(X))^2 = -(\det(C_{q,p} X))^2. \tag{5.4.6}$$

Notice that $(\det(X))^2 > 0$, whereas $-(\det(C_{q,p} X)^2 \leq 0$, a contradiction. Analogously one can derive that $\det(D_q) \neq 0$. Therefore, the four components of $\mathrm{O}(p,q)$ can be written as:

- $\mathrm{O}_+^\uparrow(p,q) = \{T \in \mathrm{O}(p,q);\ \det(A_p) > 0, \det(D_q) > 0\}$,
- $\mathrm{O}_-^\uparrow(p,q) = \{T \in \mathrm{O}(p,q);\ \det(A_p) > 0, \det(D_q) < 0\}$,
- $\mathrm{O}_+^\downarrow(p,q) = \{T \in \mathrm{O}(p,q);\ \det(A_p) < 0, \det(D_q) > 0\}$,
- $\mathrm{O}_-^\downarrow(p,q) = \{T \in \mathrm{O}(p,q);\ \det(A_p) < 0, \det(D_q) < 0\}$.

Furthermore, the following are important subsets of $\mathrm{O}(p,q)$:

1. $\mathrm{O}_+^\uparrow(p,q)$,
2. $\mathrm{O}^\uparrow(p,q) = \mathrm{O}_+^\uparrow(p,q) \cup \mathrm{O}_-^\uparrow(p,q)$,
3. $\mathrm{O}_+(p,q) = \mathrm{O}_+^\uparrow(p,q) \cup \mathrm{O}_+^\downarrow(p,q)$,
4. $\mathrm{O}_+^\uparrow(p,q) \cup \mathrm{O}_-^\downarrow(p,q)$.

Finally, it is straightforward to realize that

$$\mathrm{SO}_+(p,q) = \mathrm{SO}^{\uparrow}(p,q) = \mathrm{SO}^{\uparrow}_+(p,q) = \mathrm{O}_+(p,q) \cap \mathrm{O}^{\uparrow}(p,q). \tag{5.4.7}$$

5.5 Appendix E - Cartan-Dieudonné Theorem

Theorem (Cartan-Dieudonné). *Any orthogonal transformation T in a finite dimensional quadratic space (V,g) can be expressed as a finite product of symmetries on non-isotropic hyperplanes.*

Proof. Let $\mathbf{u} \in V$ such that $g(\mathbf{u},\mathbf{u}) \neq 0$ and let $U = \{\alpha\mathbf{u} : \alpha \in \mathbb{R}\}$. Let $\mathbf{v} \in V$ and write it as $\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp}$ such that $g(\mathbf{v}_{\perp},\mathbf{u}) = 0$. We know that

$$\mathbf{v}_{\perp} = \mathbf{v} - \frac{g(\mathbf{v},\mathbf{u})}{g(\mathbf{u},\mathbf{u})}\mathbf{u}. \tag{5.5.1}$$

Notice that since $\det(S_U) = -1$, the application is a reflection. Moreover,

$$\begin{aligned} S_U(\mathbf{v}) &= -\mathbf{v}_{||} + \mathbf{v}_{\perp} \\ &= -\frac{g(\mathbf{v},\mathbf{u})}{g(\mathbf{u},\mathbf{u})}\mathbf{u} + \mathbf{v} - \frac{g(\mathbf{v},\mathbf{u})}{g(\mathbf{u},\mathbf{u})}\mathbf{u} \\ &= \mathbf{v} - 2\frac{g(\mathbf{v},\mathbf{u})}{g(\mathbf{u},\mathbf{u})}\mathbf{u}. \end{aligned} \tag{5.5.2}$$

Now, notice that for any two vectors $\mathbf{v},\mathbf{u} \in V$ such that $g(\mathbf{u},\mathbf{u}) = g(\mathbf{v},\mathbf{v}) \neq 0$, one can associate them by at most two reflections. Indeed, suppose that $g(\mathbf{v}-\mathbf{u},\mathbf{v}-\mathbf{u}) \neq 0$. Then,

$$\begin{aligned} S_{\mathbf{u}-\mathbf{v}}(\mathbf{v}) &= \mathbf{v} - 2\frac{g(\mathbf{v},\mathbf{v}-\mathbf{u})}{g(\mathbf{v}-\mathbf{u},\mathbf{v}-\mathbf{u})}(\mathbf{v}-\mathbf{u}) \\ &= \mathbf{v} - (\mathbf{v}-\mathbf{u}) = \mathbf{u}. \end{aligned} \tag{5.5.3}$$

Otherwise, $g(\mathbf{v}-\mathbf{u},\mathbf{v}-\mathbf{u}) = 0$. That yields $g(\mathbf{v},\mathbf{v}) = g(\mathbf{u},\mathbf{u}) = g(\mathbf{v},\mathbf{u}) \neq 0$, which implies that $g(\mathbf{v}+\mathbf{u},\mathbf{v}+\mathbf{u}) \neq 0$. Therefore,

$$\begin{aligned} S_{\mathbf{u}+\mathbf{v}}(\mathbf{u}) &= \mathbf{v} + \mathbf{u} - 2\frac{g(\mathbf{u},\mathbf{v}+\mathbf{u})}{g(\mathbf{v}+\mathbf{u},\mathbf{v}+\mathbf{u})}(\mathbf{v}+\mathbf{u}) \\ &= \mathbf{v} - (\mathbf{v}+\mathbf{u}) = -\mathbf{u}, \end{aligned} \tag{5.5.4}$$

which finally yields:

$$S_{\mathbf{u}}S_{\mathbf{u}+\mathbf{v}}(\mathbf{v}) = S_{\mathbf{u}}(-\mathbf{u}) = -\mathbf{u} - 2\frac{g(-\mathbf{u},\mathbf{u})}{g(\mathbf{u},\mathbf{u})}\mathbf{u} = \mathbf{u}. \tag{5.5.5}$$

We proceed by induction on $\dim(V) = n$. It is clear that the affirmation is true for $\dim(V) = 1$. Suppose it is true for $\dim(V) = n$ and let T be an isometry. Then, consider a vector space V such that $\dim V = n+1$ and let $\mathbf{v} \in V$ such that $g(\mathbf{v},\mathbf{v}) \neq 0$. Since T is an isometry, $g(T(\mathbf{v}), T(\mathbf{v})) \neq 0$ and therefore, as we have seen, it is possible to associate $\mathbf{v}$ and $T(\mathbf{v})$ by at most 2 reflections, let us say

$$(S \circ T)(\mathbf{v}) = \mathbf{v}. \tag{5.5.6}$$

Since $S^{-1} = S$, we get $T(\mathbf{v}) = S(\mathbf{v})$, where S are 2 reflections. This caracterization of T can be expanded to all of $U = \text{span}(\mathbf{v})$. Now, for $U^{\perp}$, since $\dim U^{\perp} = n$, by the induction hypothesis it follows that there is a finite number of reflections, let us say Σ, such that

$$T(\mathbf{u}) = \Sigma(\mathbf{u}). \tag{5.5.7}$$

Then, it follows that T is always written as a finite sum of reflections through non-isotropic hyperplanes. □

5.6 Appendix F - Clifford-Lipschitz Group Theorem

Theorem. *Let $\mathcal{C}\ell_{p,q}$ be the Clifford algebra with respect to the quadratic space $\mathbb{R}^{p,q}$ and let $\sigma : \Gamma_{p,q} \to Aut(\mathcal{C}\ell_{p,q})$ and $\hat{\sigma} : \Gamma_{p,q} \to Aut(\mathcal{C}\ell p,q)$ be defined as above. Also, let $\hat{\Gamma}^{+}_{p,q} = \hat{\Gamma}_{p,q} \cap \mathcal{C}\ell^{+}_{p,q}$. Then,*

$$\begin{aligned}
&\sigma(\Gamma_{p,q}) = O(p,q), \text{ if } n = p+q \text{ is even},\\
&\sigma(\Gamma_{p,q}) = SO(p,q), \text{ if } n = p+q \text{ is odd},\\
&\hat{\sigma}(\hat{\Gamma}_{p,q}) = O(p,q),\\
&\hat{\sigma}(\hat{\Gamma}^{+}_{p,q}) = SO(p,q).
\end{aligned} \tag{5.6.1}$$

Proof. The application $\sigma : \Gamma_{p,q} \to \text{Aut}(\mathcal{C}\ell_{p,q})$ is defined by $\sigma(a)(x) = axa^{-1}$. Notice that

$$
\begin{aligned}
2g(\sigma(a)(\mathbf{v}),\sigma(a)(\mathbf{u})) &= \sigma(a)(\mathbf{v})\sigma(a)(\mathbf{u}) + \sigma(a)(\mathbf{u})\sigma(a)(\mathbf{v}) \\
&= a\mathbf{v}a^{-1}a\mathbf{u}a^{-1} + a\mathbf{u}a^{-1}a\mathbf{v}a^{-1} \\
&= a\mathbf{v}\mathbf{u}a^{-1} + a\mathbf{u}\mathbf{v}a^{-1} \\
&= a(\mathbf{v}\mathbf{u}+\mathbf{u}\mathbf{v})a^{-1} = a(2g(\mathbf{v},\mathbf{u}))a^{-1} \\
&= 2g(\mathbf{v},\mathbf{u}),
\end{aligned} \tag{5.6.2}
$$

that is,

$$
g(\sigma(a)(\mathbf{v}),\sigma(a)(\mathbf{u})) = g(\mathbf{v},\mathbf{u}), \tag{5.6.3}
$$

which implies that $\sigma(a) \in \mathrm{O}(p,q)$. Thus, we can write $\sigma : \Gamma_{p,q} \to \mathrm{O}(p,q)$. Furthermore,

$$
\sigma(ab)(\mathbf{v}) = ab\mathbf{v}(ab)^{-1} = ab\mathbf{v}b^{-1}a^{-1} = \sigma(a)\sigma(b)(\mathbf{v}), \tag{5.6.4}
$$

which means that, since $\mathbf{v}$ is arbitrary, $\sigma(ab) = \sigma(a)\sigma(b)$, that is, σ is a homomorphism. Now, we know that the determinant of a linear transformation can be written in terms of the exterior product. In fact, for a orthonormal basis $\{\mathbf{e}_i\}$ we can write

$$
\sigma(a)(\mathbf{e}_1) \wedge \cdots \wedge \sigma(a)(\mathbf{e}_n) = (\det(\sigma(a)))\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n. \tag{5.6.5}
$$

Moreover, for vectors $\mathbf{u},\mathbf{v}$ it follows that

$$
\begin{aligned}
\sigma(a)(\mathbf{v}) \wedge \sigma(a)(\mathbf{u}) &= (a\mathbf{v}a^{-1}) \wedge (a\mathbf{u}a^{-1}) \\
&= \frac{1}{2}(a\mathbf{v}a^{-1}a\mathbf{u}a^{-1} - a\mathbf{u}a^{-1}a\mathbf{v}a^{-1}) \\
&= a(\mathbf{v}\wedge\mathbf{u})a^{-1}.
\end{aligned} \tag{5.6.6}
$$

Using Eq.(5.6.5) and Eq.(5.6.6), there holds

$$
a(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n)a^{-1} = \det(\sigma(a))(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n). \tag{5.6.7}
$$

Taking $a_{[k]} \in \Lambda^k(V)$, it follows that

$$
\begin{aligned}
a_{[k]}(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n) &= a_{[k]_\flat} \rfloor (\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n) \\
&= (-1)^{k(n-1)}(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n) \lfloor a_{[k]_\flat} \\
&= (-1)^{k(n-1)}(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n)a_{[k]}.
\end{aligned} \tag{5.6.8}
$$

Now, if $n = p + q$ is odd, $n-1$ is even and, therefore,

$$a_{[k]}(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n) = (\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n)a_{[k]}. \tag{5.6.9}$$

One can generalize the result for the multivector $a \in \Gamma_{p,q}$. Using that n is odd yields

$$a(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n)a^{-1} = (\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n), \tag{5.6.10}$$

which by Eq.(5.6.5) means that

$$\det \sigma(a) = 1. \tag{5.6.11}$$

For n even, $n-1$ is odd and therefore one can conclude that

$$a(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n)a^{-1} = (-1)^k(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n),$$

which means that $\det(\sigma(a)) = \pm 1$. Namely,

$$\begin{aligned} \det \sigma(a) &= 1 \quad \text{if} \quad a \in \mathcal{C}\ell_{p,q}^{+}, \\ \det \sigma(a) &= -1 \quad \text{if} \quad a \in \mathcal{C}\ell_{p,q}^{-}. \end{aligned} \tag{5.6.12}$$

Finally,

$$\begin{aligned} \sigma(\Gamma_{p,q}) &\subset \mathrm{O}(p,q) \quad (n \text{ even}), \\ \sigma(\Gamma_{p,q}) &\subset \mathrm{SO}(p,q) \quad (n \text{ odd}). \end{aligned} \tag{5.6.13}$$

For the twisted adjoint representation, we have that

$$\hat{\sigma}(a)(\mathbf{v}) = \hat{a}\mathbf{v}a^{-1}. \tag{5.6.14}$$

Using similar computations, we can take $a_{[k]} \in \Lambda^k(\mathbb{R}^{p,q})$ and, analogously to Eq.(5.6.6), for vectors $\mathbf{v}_1, \ldots, \mathbf{v}_i$, it is possible to write that

$$\begin{aligned} \hat{\sigma}(a_{[k]})(\mathbf{v}_1) \wedge \cdots \wedge \hat{\sigma}(a_{[k]})(\mathbf{v}_i) &= (-1)^{k(i-1)}\hat{\sigma}(a_{[k]})(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_i) \\ &= (-1)^{ik}\sigma(a_{[k]})(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_i), \end{aligned} \tag{5.6.15}$$

which implies that

$$\begin{aligned}\hat{\sigma}(a_{[k]})(\mathbf{e}_1)\wedge\cdots\wedge\hat{\sigma}(a_{[k]})(\mathbf{e}_n) &= (-1)^{nk}a_{[k]}(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_n)a_{[k]}^{-1}\\ &= (-1)^{nk}(-1)^{k(n-1)}(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_n),\end{aligned} \tag{5.6.16}$$

which finally yields

$$\det(\hat{\sigma}(a_{[k]})) = (-1)^k. \tag{5.6.17}$$

Notice the determinant no longer depends on the dimension of $\mathbb{R}^{p,q}$, only on the parity of k. That way, one can write

$$\hat{\sigma}(\Gamma_{p,q}) \subset \mathrm{O}(p,q). \tag{5.6.18}$$

Furthermore, defining $\Gamma_{p,q}^+ = \Gamma_{p,q} \cap \mathcal{C}\ell_{p,q}^+$, it follows that

$$\hat{\sigma}(\Gamma_{p,q}^+) \subset \mathrm{SO}(p,q). \tag{5.6.19}$$

The application $\hat{\sigma}$ is surjective in $\mathrm{O}(p,q)$. Indeed, from Eq.(2.1.11) we know that a reflection with respect to the orthogonal hyperplane of $\mathbf{u} \in \mathbb{R}^{p,q}$ is given by

$$\hat{\sigma}(\mathbf{u}) = S_{\mathbf{u}}. \tag{5.6.20}$$

Now, according to Cartan-Dieudonné theorem, any $T \in \mathrm{O}(p,q)$ can be expressed as a finite number of reflections by non-isotropic vectors. Namely, there is a finite number of vectors $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ such that $g(\mathbf{u}_i,\mathbf{u}_i) \neq 0$ (and therefore, there exists $\mathbf{u}_i^{-1} \in \mathcal{C}\ell_{p,q}^*$ for each $i \in \{1,\dots,k\}$) for which the following holds:

$$T = \hat{\sigma}(\mathbf{u}_1)\dots\hat{\sigma}(\mathbf{u}_k) = \hat{\sigma}(\mathbf{u}_1\dots\mathbf{u}_k), \tag{5.6.21}$$

which shows that $\hat{\sigma}$ is surjective. It follows immediately that the restriction $\hat{\sigma}_{\Gamma_{p,q}^+}$ and σ are also surjective, which completes the proof. □

5.7 Appendix G - Simple Algebras

An important definition in the study of Clifford algebras is the notion of division algebras:

Definition 5.2. *An algebra $\mathcal{A}$ is called a division algebra if for every $a,b \in A$ such that b is non-zero, there are unique $x,y \in \mathcal{A}$ such that $a = bx$ and $a = yb$.*

It is known that every Clifford algebra can be expressed either by $\mathbb{K}\otimes\mathcal{M}(n,\mathbb{R})$ or by $[\mathbb{K}\otimes\mathcal{M}(n,\mathbb{R})]\oplus[\mathbb{K}\otimes\mathcal{M}(n,\mathbb{R})]$, for $\mathbb{K}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$, which are division algebras. One can prove the following proposition:

Proposition 5.3. *Every algebra $\mathcal{A}=\mathbb{K}\otimes\mathcal{M}(n,\mathbb{R})$ is a simple algebra.*

Proof. Let $I\subset\mathcal{A}$ be an ideal and let $x\in I$. There is a set of real matrices $\{E_{AB}\}$ (capital letters denote natural numbers from 1 to n) where each E_{AB} is a $n\times n$ matrix such that all their entries are 0 but the entry AB. This can be equivalently stated as

$$(E_{AB})_{CD}=\delta_{AC}\delta_{BD}. \tag{5.7.1}$$

These matrices clearly satisfy the product rule $E_{AB}E_{CD}=\delta_{BC}E_{AD}$. Moreover, this set forms a basis for $\mathcal{M}(n,\mathbb{R})$. Then, x can be written as $x=\sum_{AB}x_{AB}E_{AB}$, where $x_{AB}\in\mathbb{K}$. We suppose $x\neq 0$ and therefore there is a $x_{AB}\neq 0$. It follows that

$$\begin{aligned}\sum_{C}E_{CA}xE_{BC}&=\sum_{CDE}x_{DE}E_{CA}E_{DE}\\&=\sum_{CDE}x_{DE}\delta_{AD}\delta_{BE}E_{CC}=x_{AB}\sum_{C}E_{CC}=x_{AB}1,\end{aligned} \tag{5.7.2}$$

where 1 is the identity matrix. Since $x_{AB}\in\mathbb{K}$ is not zero, and since $\mathbb{K}$ is an associative division algebra, there is $x_{AB}^{-1}\in\mathbb{K}$. Therefore,

$$\sum_{C}x_{AB}^{-1}E_{CA}xE_{BC}=1, \tag{5.7.3}$$

which implies that $1\in I$. Therefore, for every $a\in\mathcal{A}$, we have $a1=a\in I$, concluding that $I=\mathcal{A}$. Otherwise, $I=\{0\}$, and we are done. Then, an algebra $\mathcal{A}=\mathbb{K}\otimes\mathcal{M}(n,\mathbb{R})$ is a simple algebra. □

It follows that every Clifford algebra is either a simple algebra or a direct sum of simple algebras. Now, let $\mathcal{A}$ be a simple algebra. We shall see that it must be written as $\mathcal{A}=\mathbb{K}\otimes\mathcal{M}(n,\mathbb{R})$.

Definition 5.4. *Let $\mathcal{A}$ be an algebra with the product denoted by juxtaposition. An non-null element $f\in\mathcal{A}$ is called an idempotent if $f^2=f$.*

If the algebra $\mathcal{A}$ is a division algebra[2], the unique idempotent is the identity 1.

[2]A division algebra satisfies the following condition: if $ab=0$, then $a=0$ or $b=0$, for every $a,b\in\mathcal{A}$. More details in ref. [8].

Indeed, if $f^2 = f$, where $f \neq 0$, then

$$f^2 = f \Rightarrow f^2 - f = 0 \Rightarrow f(f-1) = 0, \tag{5.7.4}$$

which means that either $f = 0$ or $f - 1 = 0$. Since $f \neq 0$, it follows that $f = 1$. Now, two idempotents are called *orthogonal* if $f_1 f_2 = f_2 f_1 = 0$. An idempotent f is said to by *primitive* if it cannot be written as the sum of other two orthogonal idempotents. Consider a set of n primitive idempotents $\{f_1, \ldots, f_n\}$ in $\mathcal{A}$ which are mutually orthogonal, that is,

$$f_A f_B = \delta_{AB} f_A. \tag{5.7.5}$$

Now, define $\mathcal{A}_{AB} = f_A \mathcal{A} f_B$. For those sets, we have

$$\mathcal{A}_{AB}\mathcal{A}_{CD} = f_A \mathcal{A} f_B f_C \mathcal{A} f_D = \delta_{BC} f_a \mathcal{A} f_B \mathcal{A} f_D. \tag{5.7.6}$$

Moreover, the set $\mathcal{A} f_a \mathcal{A}$ is a two-sided ideal of $\mathcal{A}$ and since this algebra is simple, one has $\mathcal{A} f_A \mathcal{A}$ equal to either $\mathcal{A}$ or $\{0\}$. Since f_B is nonzero, it yields

$$\mathcal{A}_{AB}\mathcal{A}_C D = \delta_{BC}\mathcal{A}_{AD}. \tag{5.7.7}$$

Now, it is clear that $f_A \in \mathcal{A}_{AA}$, for each A. By the above equation, one can deduce that $f_1 \in \mathcal{A}_{11} = \mathcal{A}_{1A}\mathcal{A}_{A1}$, for every value A. It follows that there exist $\boldsymbol{\mathcal{E}}_{1A} \in \mathcal{A}_{1A}$ and $\boldsymbol{\mathcal{E}}_{A1} \in \mathcal{A}_{1A}$ such that

$$f_1 = \boldsymbol{\mathcal{E}}_{1A}\boldsymbol{\mathcal{E}}_{A1}. \tag{5.7.8}$$

Now, we choose $\boldsymbol{\mathcal{E}}_{1A}, \boldsymbol{\mathcal{E}}_{A1}$ such that

$$f_B \boldsymbol{\mathcal{E}}_{A1} = \delta_{AB}\boldsymbol{\mathcal{E}}_{A1}, \quad \boldsymbol{\mathcal{E}}_{1A} f_B = \delta_{AB}\boldsymbol{\mathcal{E}}_{1A}. \tag{5.7.9}$$

Furthermore, we shall define the quantities $\boldsymbol{\mathcal{E}}_{AB}$ as

$$\boldsymbol{\mathcal{E}}_{AB} = \boldsymbol{\mathcal{E}}_{A1}\boldsymbol{\mathcal{E}}_{1B}, \tag{5.7.10}$$

which immediately yields

$$f_C = \boldsymbol{\mathcal{E}}_{AB} = \delta_{AC}\boldsymbol{\mathcal{E}}_{AB}, \quad \boldsymbol{\mathcal{E}}_{AB} f_C = \delta_{BC}\boldsymbol{\mathcal{E}}_{AB}. \tag{5.7.11}$$

Notice that with the above definition, one can work out the following:

$$\begin{aligned}\boldsymbol{\mathcal{E}}_{AB}\boldsymbol{\mathcal{E}}_{CD} &= \boldsymbol{\mathcal{E}}_{A1}\boldsymbol{\mathcal{E}}_{1B}\boldsymbol{\mathcal{E}}_{C1}\boldsymbol{\mathcal{E}}_{1D} = \delta_{BC}\boldsymbol{\mathcal{E}}_{A1}\boldsymbol{\mathcal{E}}_{1B}\boldsymbol{\mathcal{E}}_{B1}\boldsymbol{\mathcal{E}}_{1D} \\ &= \delta_{BC}\boldsymbol{\mathcal{E}}_{A1}f_1\boldsymbol{\mathcal{E}}_{1D} = \delta_{BC}\boldsymbol{\mathcal{E}}_{A1}\boldsymbol{\mathcal{E}}_{1D},\end{aligned} \tag{5.7.12}$$

which more concisely reads

$$\boldsymbol{\mathcal{E}}_{AB}\boldsymbol{\mathcal{E}}_{CD} = \delta_{BC}\boldsymbol{\mathcal{E}}_{AD}. \tag{5.7.13}$$

Making a comparison, these are the same relations that rule the basis $\{E_{AB}\}$ presented before. In fact, it is straightforward that $\{\boldsymbol{\mathcal{E}}_{AB}\}$ can be seen (by an isomorphism) as a basis for the space of $n \times n$ matrices. Also, we have that

$$\sum_A \boldsymbol{\mathcal{E}}_{AA} = 1_{\mathcal{A}}, \tag{5.7.14}$$

where $1_{\mathcal{A}}$ is the unity of the algebra $\mathcal{A}$. Indeed, notice that

$$1_{\mathcal{A}}\boldsymbol{\mathcal{E}}_{BC} = \sum_A \boldsymbol{\mathcal{E}}_{AA}\boldsymbol{\mathcal{E}}_{BC} = \sum_A \delta_{AB}\boldsymbol{\mathcal{E}}_{AC} = \boldsymbol{\mathcal{E}}_{BC}. \tag{5.7.15}$$

Now, it is clear by Eq.(5.7.13) that $\boldsymbol{\mathcal{E}}_{AA}$ is an idempotent. Moreover, notice that the set $\mathcal{A}_{AA} = f_A\mathcal{A}f_A$ is an ideal of $\mathcal{A}$. For $x \in \mathcal{A}_{AA}$, it follows that $xf_A = x$ and $f_Ax = x$, which means that f_A is a unity in A_{AA}. One can then prove the following proposition:

Proposition 5.5. *An element f_A is primitive if and only if f_A is the unique idempotent in $\mathcal{A}_{AA}$.*

Proof. Suppose f_A is not primitive. Then, we can write $f_A = g_A + h_A$, where $h_Ag_A = g_Ah_A = 0$. The last equation means that neither g_A nor h_A is equal to f_A, since $f_Af_A = f_A \neq 0$. Now, it follows that $f_Ag_A = g_Af_A = g_A$ and $f_Ah_A = h_Af_A = h_A$, which means that $g_A, h_A \in \mathcal{A}_{AA}$. Therefore, f_A is not the unique idempotent in $\mathcal{A}_{AA}$

For the reverse condition, suppose there is another idempotent $g_A \in \mathcal{A}_{AA}$ not equal to f_A. Then, $f_A - g_A$ is another idempotent. Indeed, first notice that $f_Ag_A = g_Af_A = g_A$. Then,

$$(f_A - g_A)(f_A - g_A) = f_A + f_Ag_A - f_Ag_A + g_A = f_A - g_A. \tag{5.7.16}$$

Moreover, g_A and $f_A - g_A$ are orthogonal, since $(f_A - g_A)g_A = g_A(f_A - g_A) = g_Af_A - g_A = 0$. Finally, notice that $f_A = (f_A - g_A) + g_A$. This implies that f_A is not primitive, which concludes the proof. □

By the proposition, since $\boldsymbol{\mathcal{E}}_{AA} \in f_A \mathcal{A} f_A$ and f_A is primitive, it follows that

$$\boldsymbol{\mathcal{E}}_{AA} = f_A. \tag{5.7.17}$$

Proposition 5.6. *If $\mathcal{A}$ is associative, then $\mathcal{A}_{AA} = f_A \mathcal{A} f_A \simeq \mathbb{K}$ is a division algebra.*

Proof. Let $I_A = \mathcal{A} f_A$ be a left ideal of $\mathcal{A}$. Since f_A is primitive, it follows that I_A is a minimal ideal. Let now J_A be a non-null left ideal of $\mathcal{A}_{AA}$. It is clear that $J_A \subset \mathcal{A}_{AA}$ and that

$$\mathcal{A}_{AA} J_A \subset \mathcal{A} f_A \mathcal{A} f_A = f_A \mathcal{A} I_A, \tag{5.7.18}$$

that is, $\mathcal{A} J_A \subset I_A$. Nevertheless, by the minimality of I_A and since $\mathcal{A} J_A$ is a left ideal of $\mathcal{A}$, one has $\mathcal{A} J_A = I_A$. On the other hand, one can see that

$$\mathcal{A}_{AA} = f_A \mathcal{A} f_A = f_A I_A = f_A \mathcal{A} J_A \subset f_A \mathcal{A} f_A \mathcal{A} f_A J_A \subset J_A, \tag{5.7.19}$$

wherein we used that $f_A \mathcal{A} f_A J_A = J_A$. Therefore, $\mathcal{A}_{AA} = J_A$ and it follows
Consider now $z \in \mathcal{A}_{AA}$ such that $z \neq 0$. It suffices to notice that $\mathcal{A}_{AA} z$ is a non-null left ideal of $\mathcal{A}_{AA}$. Therefore, there is a $z' \in \mathcal{A}_{AA}$ such that

$$zz' = f_A, \tag{5.7.20}$$

where f_A was proved to be the unity in $\mathcal{A}_{AA}$. The same idea can be implemented for the right product, and the result follows. □

Proposition 5.7. *$\mathcal{A}_{AA} \simeq \mathcal{A}_{BB}$ for all $A, B \in \{1, \ldots, n\}$.*

Proof. Define a function $\phi_{AB} : \mathcal{A}_{AA} \to \mathcal{A}_{BB}$ by

$$\phi_{AB}(x_A) = \boldsymbol{\mathcal{E}}_{BA} x_A \boldsymbol{\mathcal{E}}_{AB}, \tag{5.7.21}$$

which is clearly a linear application. Using the equalities derived for $\boldsymbol{\mathcal{E}}_{AB}$ one can see that

$$\begin{aligned} \phi_{AB}(x_A y_A) &= \boldsymbol{\mathcal{E}}_{BA} x_A y_A \boldsymbol{\mathcal{E}}_{AB} = \boldsymbol{\mathcal{E}}_{BA} x_A f_A y_A \boldsymbol{\mathcal{E}}_{AB} \\ &= \boldsymbol{\mathcal{E}}_{BA} x_A \boldsymbol{\mathcal{E}}_{AB} \boldsymbol{\mathcal{E}}_{BA} y_A \boldsymbol{\mathcal{E}}_{AB} = \phi_{AB}(x_A) \phi_{AB}(y_A). \end{aligned} \tag{5.7.22}$$

Therefore, ϕ_{AB} is an algebra homomorphism. Furthermore, notice that $\phi_{AB}^{-1} = \phi_{BA}$ since

$$\phi_{AB}(\phi_{BA}(x_A)) = \phi_{AB}(\boldsymbol{\mathcal{E}}_{BA} x_A \boldsymbol{\mathcal{E}}_{AB}) = \boldsymbol{\mathcal{E}}_{AB} \boldsymbol{\mathcal{E}}_{BA} x_A \boldsymbol{\mathcal{E}}_{AB} \boldsymbol{\mathcal{E}}_{BA} = f_A x_A f_A = x_A, \tag{5.7.23}$$

which concludes the proof. □

One can, for instance, define the division algebra $\mathbb{K}$ by taking $x_1 \in \mathcal{A}_{11}$ and $x_A = \mathcal{E}_{A1} x_1 \mathcal{E}_{1A}$, making

$$x = \sum_A x_A = \sum_A \mathcal{E}_{A1} x_1 \mathcal{E}_{1A} \in \mathbb{K} \tag{5.7.24}$$

One can then define a linear application $p : \mathbb{K} \to \mathcal{A}_{11}$ such that

$$p(x) = \mathcal{E}_{11} x \mathcal{E}_{11}, \tag{5.7.25}$$

and it is clear that the inverse $p^{-1} : \mathcal{A}_{11} \to \mathbb{K}$ is given by

$$p^{-1}(x_1) = \sum_A \mathcal{E}_{A1} x_1 \mathcal{E}_{1A}, \tag{5.7.26}$$

since $\mathcal{E}_{11}\mathcal{E}_{A_1} = \delta_{1A}\mathcal{E}_{11} = \mathcal{E}_{1A}\mathcal{E}_{11}$. Therefore,

$$\mathbb{K} \simeq f_A \mathcal{A} f_A, \tag{5.7.27}$$

for every primitive idempotent f_A.

Additionally, one can notice that $x \in \mathbb{K}$ commutates with every element in $\{\mathcal{E}_{AB}\}$. Indeed,

$$\begin{aligned} x\mathcal{E}_{AB} &= \sum_C \mathcal{E}_{C1} x_1 \mathcal{E}_{1C} \mathcal{E}_{AB} = \mathcal{E}_{A1} x_1 \mathcal{E}_{1A} \mathcal{E}_{AB} \\ &= \mathcal{E}_{A1} x_1 \mathcal{E}_{1B} = \mathcal{E}_{AB} \mathcal{E}_{B1} x_1 \mathcal{E}_{1B} \\ &= \sum_C \mathcal{E}_{AB} \mathcal{E}_{C1} x_1 \mathcal{E}_{1C} = \mathcal{E}_{AB} x. \end{aligned} \tag{5.7.28}$$

Let now $x \in \mathcal{A}$. Since $1_{\mathcal{A}} = \sum_A f_A = \sum_A \mathcal{E}_{AA}$, it follows that

$$\begin{aligned} x &= \sum_A \mathcal{E}_{AA} x \sum_B \mathcal{E}_{BB} = \sum_{AB} \mathcal{E}_{AA} x \mathcal{E}_{BA} \mathcal{E}_{AB} \\ &= \sum_{ABC} \mathcal{E}_{CA} x \mathcal{E}_{BC} \mathcal{E}_{AB} \\ &= \sum_{ABC} (x_C)_{AB} \mathcal{E}_{AB}, \end{aligned} \tag{5.7.29}$$

where one defines $(x_C)_{AB} = \mathcal{E}_{CA} x \mathcal{E}_{BC} \in \mathcal{A}_{CC}$. Now, it follows that

$$(x_D)_{AB} = \mathcal{E}_{DC}(x_C)\mathcal{E}_{CD}. \tag{5.7.30}$$

One can then make $x_{AB} = \sum_C (x_C)_{AB}$, which implies that

$$x_{AB} = \sum_C \boldsymbol{\mathcal{E}}_{CA} x \boldsymbol{\mathcal{E}}_{BC}, \tag{5.7.31}$$

which yields in Eq.(5.7.29) the equality

$$x = \sum_{AB} x_{AB} \boldsymbol{\mathcal{E}}_{AB}, \tag{5.7.32}$$

where $x_{AB} \in \mathbb{K}$ and $\{\boldsymbol{\mathcal{E}}_{AB}\}$ is a basis for the $n \times n$ matrices. Therefore, one can conclude that if $\mathcal{A}$ is a simple algebra, it is written as $\mathbb{K} \otimes \mathcal{M}(n, \mathbb{R})$. That means there is a representation ρ such that

$$\rho(x) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & \ddots & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}, \tag{5.7.33}$$

in which the representation space is $\mathcal{A} f_1$. This is due to the choice of f_1 when defining $\boldsymbol{\mathcal{E}}_{AB} = \boldsymbol{\mathcal{E}}_{A1} \boldsymbol{\mathcal{E}}_{1B}$. Then, from $\mathcal{A} \simeq \mathbb{K} \otimes \mathcal{M}(n, \mathbb{R})$ it is clear that for each choice of f_A we have

$$\mathcal{A} f_A = \mathbb{K} \otimes \mathbb{R}. \tag{5.7.34}$$

Finally, one can prove the following result:

Proposition 5.8. *Let $\mathcal{A}$ be an simple unital algebra and let $I \subset \mathcal{A}$ be a non-trivial minimal left ideal. Then, there is an idempotent $f \in I$ such that $I = \mathcal{A} f$.*

Proof. It is straightforward to see that $\mathrm{Ann}(I) = \{a \in \mathcal{A} : az = 0, \forall z \in I\}$ is a two-sided ideal in $\mathcal{A}$. Since $\mathcal{A}$ is simple, it follows that $\mathrm{Ann}(I) = \{0\}$ or $\mathcal{A}$. If $\mathrm{Ann}(I) = \mathcal{A}$, then for every $z \in I$ and $a \in \mathcal{A}$ there holds $az = 0$. In particular, $a = 1$ yields $z = 0$, implying that $I = \{0\}$, which by hypothesis is absurd. Then, $\mathrm{Ann}(I) = \{0\}$ and it follows that there must be $x \in I$, $x \neq 0$ such that $Ix \neq \{0\}$. Since Ix is a left ideal and $Ix \subset I$, since I is minimal it follows that $Ix = I$. Now, because $x \in I$, then there must be $f \in I$, $f \neq 0$ such that $fx = x$, which implies that

$$(f^2 - f)x = 0. \tag{5.7.35}$$

Since $\mathrm{Ann}(I) = \{0\}$, it follows that $f^2 = f$. Furthermore, since $f \in I$, then $\mathcal{A} f \subset I$. In particular, $f \neq 0$, then $\mathcal{A}f \neq \{0\}$. By minimality of I, it follows that

$$\mathcal{A}f = I. \tag{5.7.36}$$

□

5.8 Appendix H - $\mathrm{SO}_+(1,3) \simeq \mathrm{SL}(2,\mathbb{C})$

Consider the following construction: it is known that one can make a correspondence from $\mathbb{R}^{1,3}$ to the set of hermitian 2×2 matrices, that is, to

$$\mathcal{H} = \{A \in M(2,\mathbb{C}) : A = A^\dagger\}, \tag{5.8.1}$$

where $A^\dagger = \overline{A^T}$. Indeed, let $\mathbf{u} \in \mathbb{R}^{1,3}$ such that $\mathbf{u} = (t,x,y,z)$. Then, since the Pauli matrices σ_μ ($\mu = 0,\dots,3$) are a basis for $\mathcal{H}$, one can write

$$U = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \in \mathcal{H}. \tag{5.8.2}$$

Now, $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathcal{H}$ by conjugation, that is, for every $g \in \mathrm{SL}(2,\mathbb{C})$ and $M \in \mathcal{H}$ we have the representation

$$\rho(g)M = gMg^{-1}, \tag{5.8.3}$$

which is indeed a representation since

$$\begin{aligned} \rho(g)\rho(g')M &= \rho(g)(g'Mg'^{-1}) \\ &= (gg')M(g'^{-1}g^{-1}) \\ &= \rho(gg')M. \end{aligned} \tag{5.8.4}$$

Now, notice that ρ preserves the metric in $\mathbb{R}^{1,3}$. Indeed, letting U be the matrix in Eq.(5.8.2), it follows that

$$\begin{aligned} \det(\rho(g)U) &= \det(gUg^{-1}) \\ &= \det(g)\det(U)\det(g^{-1}) \\ &= \det(U) \\ &= t^2 - x^2 - y^2 - z^2, \end{aligned} \tag{5.8.5}$$

Therefore, it follows that ρ is a homomorphism such that

$$\rho : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{O}(1,3). \tag{5.8.6}$$

Now, since ρ is a homomorphism, it maps the identity element of $\mathrm{SL}(2,\mathbb{C})$ to the one in $\mathrm{O}(1,3)$. Furthermore, since $\mathrm{SL}(2,\mathbb{C})$ is connected, the image of ρ, being a continuous map, must also be connected. Therefore, it must be the identity component of $\mathrm{O}(p,q)$, that is:

$$\rho : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}_+(1,3). \tag{5.8.7}$$

Now, it is clear that ρ is *at least* a two-to-one map, since

$$\rho(g) = \rho(-g). \tag{5.8.8}$$

Moreover, it is *exactly* a two-to-one map. Notice that if $\rho(g) = \rho(h)$, then it follows that if $M \in \mathcal{H}$ then

$$\begin{aligned} &\rho(g)M = \rho(h)M \\ \Longleftrightarrow\ &gMg^{-1} = hMh^{-1} \\ \Longleftrightarrow\ &(h^{-1}g)M = M(h^{-1}g), \end{aligned} \tag{5.8.9}$$

which means $h^{-1}g$ is commutative with respect to any matrix in $\mathcal{H}$. But notice that this means $h^{-1}g$ is commutative to the Pauli matrices, since they form the basis of the space. Thus, we proceed to show that $h^{-1}g = \pm 1$, the identity matrix in $\mathrm{SL}(2,\mathbb{C})$. First, let

$$h^{-1}g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.8.10}$$

Since $h^{-1}g$ commutates with the Pauli matrices, we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5.8.11}$$

which implies that $a = d$ and $b = c$. Changing $h^{-1}g$ accordingly we can continue writing

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}, \tag{5.8.12}$$

which yields $b = -b$, that is, $b = 0$. Therefore,

$$h^{-1}g = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}. \tag{5.8.13}$$

Since $h^{-1}g \in \mathrm{SL}(2,\mathbb{C})$, we know that

$$\det(h^{-1}g) = 1, \tag{5.8.14}$$

which translates to $a^2 = 1$. Finally, we have that

$$h^{-1}g = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.8.15}$$

This means that $g = \pm h$. In other words, ρ is a two-to-one map. Altogether, we can write

$$\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \simeq \mathrm{SO}_+(1,3), \tag{5.8.16}$$

which implies, according to what has been presented in Chapter 2:

$$\mathrm{SL}(2,\mathbb{C}) \simeq Spin_+(1,3). \tag{5.8.17}$$

5.9 Appendix I - Fierz Identities

Aiming to shorten our notation, let

$$\psi = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \quad \psi^\dagger = \begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix}, \tag{5.9.1}$$

where

$$x = \psi_1, \quad x' = \psi_1^*, \tag{5.9.2}$$
$$y = \psi_2, \quad y' = \psi_2^*, \tag{5.9.3}$$
$$z = \psi_3, \quad z' = \psi_3^*, \tag{5.9.4}$$
$$w = \psi_4, \quad w' = \psi_4^*. \tag{5.9.5}$$

We proceed to show that

$$\mathbf{J}^2 = \Omega_1^2 + \Omega_2^2. \tag{5.9.6}$$

First, we can write

$$\Omega_1^2 = (\psi^\dagger \gamma_0 \psi)^2 = (xx' + yy' - zz' - ww')^2. \tag{5.9.7}$$

On the other hand, notice that

$$\gamma^{0123} = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{5.9.8}$$

which implies that

$$\Omega_2^2 = (\psi^\dagger \gamma_0 \gamma^{0123} \psi)^2 = -(x'z - y'w + z'x - w'y)^2. \tag{5.9.9}$$

We now follow to find $\mathbf{J}^2 = (J_0)^2 - (J_1)^2 - (J_2)^2 - (J_3)^2$, where

$$J_\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi. \tag{5.9.10}$$

Using our notation, it yields

$$(J_0)^2 = (xx' + yy' + zz' + ww')^2 \tag{5.9.11}$$
$$(J_1)^2 = (x'w + zy' + yz' + xw')^2 \tag{5.9.12}$$
$$(J_2)^2 = -(zy' - wx' - yz' + xw')^2 \tag{5.9.13}$$
$$(J_3)^2 = (zx' - wy' + xz' - yw')^2. \tag{5.9.14}$$

Now, let

$$\alpha = (xx' + yy'), \quad \beta = (zz' + ww') \tag{5.9.15}$$
$$\Delta = (wx' + yz'), \quad \sigma = (zy' + xw') \tag{5.9.16}$$
$$\lambda = (zx' - yw'), \quad \rho = (wy' - xz'). \tag{5.9.17}$$

That way, we can write

$$\Omega_1^2 = (\alpha - \beta)^2, \tag{5.9.18}$$

$$\Omega_2^2 = -(\lambda+\rho)^2, \tag{5.9.19}$$
$$(J_0)^2 = (\alpha+\beta)^2, \tag{5.9.20}$$
$$(J_1)^2 = (\Delta+\sigma)^2, \tag{5.9.21}$$
$$(J_2)^2 = -(\sigma-\Delta)^2, \tag{5.9.22}$$
$$(J_3)^2 = (\lambda-\rho)^2, \tag{5.9.23}$$

which yields, developing the quadratic terms:

$$\begin{aligned}\mathbf{J}^2 &= (\alpha+\beta)^2 - (\Delta+\sigma)^2 + (\sigma-\Delta)^2 - (\lambda-\rho)^2\\ &= \alpha^2 + 2\alpha\beta + \beta^2 - \Delta^2 - 2\delta\sigma - \sigma^2 + \sigma^2 - 2\sigma\Delta + \Delta^2 - \lambda^2 +\\ &\quad + 2\lambda\rho - \rho^2\\ &= \alpha^2 + 2\alpha\beta + \beta^2 - 4\Delta\sigma - \lambda^2 + 2\lambda\rho - \rho^2,\end{aligned} \tag{5.9.24}$$

whereas

$$\begin{aligned}\Omega_1^2 + \Omega_2^2 &= (\alpha-\beta)^2 - (\lambda+\rho)^2\\ &= \alpha^2 - 2\alpha\beta + \beta^2 - \lambda^2 - 2\lambda\rho - \rho^2.\end{aligned} \tag{5.9.25}$$

Therefore, Eq.(5.9.6) is satisfied if and only if

$$\Delta\sigma = \alpha\beta + \lambda\rho, \tag{5.9.26}$$

which is indeed true. Notice that

$$\Delta\sigma = xx'ww' + wzx'y' + xyz'w' + zz'yy', \tag{5.9.27}$$

whereas

$$\begin{aligned}\alpha\beta + \lambda\rho &= xx'zz' + xx'ww' + yy'zz' + yy'ww' + x'zy'w - x'xzz' +\\ &\quad - w'yy'w + w'yz'x\\ &= xx'ww' + x'y'wz + yw'z'x + z'zyy'\\ &= \Delta\sigma,\end{aligned} \tag{5.9.28}$$

which finally implies that

$$\mathbf{J}^2 = \Omega_1^2 + \Omega_2^2. \tag{5.9.29}$$

Now we prove that

$$\mathbf{J}^2 = -\mathbf{K}^2. \tag{5.9.30}$$

It suffices to show that

$$\mathbf{K}^2 = -\Omega_1^2 - \Omega_2^2, \tag{5.9.31}$$

since $\mathbf{J}^2 = \Omega_1^2 + \Omega_2^2$. Notice that

$$\mathbf{K}^2 = (K_0)^2 - (K_1)^2 - (K_2)^2 - (K_3)^2. \tag{5.9.32}$$

Using Eq.(5.9.8), one can write

$$i\gamma_{0123} = -\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{5.9.33}$$

Now, using that $K_\mu = \psi^\dagger \gamma_0 i\gamma_{0123}\gamma_\mu \psi$, it follows that

$$(K_0)^2 = (x'z + y'w + z'w + w'z)^2, \tag{5.9.34}$$
$$(K_1)^2 = (x'y + y'x + z'w + w'z)^2, \tag{5.9.35}$$
$$(K_2)^2 = -(-x'y + xy' - z'w + zw')^2, \tag{5.9.36}$$
$$(K_3)^2 = (xx' - yy' + zz' - ww')^2, \tag{5.9.37}$$
$$\tag{5.9.38}$$

and we have the same equations as before for Ω_1^2 and Ω_2^2. Now, let

$$\alpha' = (xx' - ww''), \quad \beta' = (zz' + ww'), \tag{5.9.39}$$
$$\Delta' = (x'z + y'w), \quad \sigma' = (z'x + w'y), \tag{5.9.40}$$
$$\lambda' = (x'y + z'w), \quad \rho' = (y'x + w'z), \tag{5.9.41}$$

which can be used to write

$$\Omega_1^2 = (\alpha' - \beta')^2, \tag{5.9.42}$$
$$\Omega_2^2 = -(\Delta' - \sigma')^2, \tag{5.9.43}$$
$$(K^0)^2 = (\Delta' + \sigma')^2, \tag{5.9.44}$$
$$(K^1)^2 = (\lambda' + \rho')^2, \tag{5.9.45}$$
$$(K^2)^2 = -(-\lambda' + \rho')^2, \tag{5.9.46}$$
$$(K^3)^2 = (\alpha' - \beta')^2, \tag{5.9.47}$$

yielding

$$\begin{aligned}\mathbf{K}^2 =&(K^0)^2-(K^1)^2-(K^2)^2-(K^3)^2\\ =&(\Delta'+\sigma')^2-(\lambda'+\rho')^2+(-\lambda'+\rho')^2-(\alpha'-\beta')^2\\ =&(\Delta')^2+2\Delta'\sigma'+(\sigma')^2-(\lambda')^2-2\lambda'\rho'+\\ &-(\rho')^2+(\lambda')^2-2\lambda'\rho'+(\rho')^2-(\alpha')^2+2\alpha'\beta'-(\beta')^2\\ =&(\Delta')^2+(\sigma')^2-(\alpha')^2-(\beta')^2-4\lambda'\rho'+2\Delta'\sigma'+2\alpha'\beta',\end{aligned} \tag{5.9.48}$$

whereas

$$\begin{aligned}-\Omega_1^2-\Omega_2^2 =&-(\alpha'+\beta')^2+(\Delta'-\sigma')^2\\ =&-(\alpha')^2-2\alpha'\beta'-(\beta')^2+(\Delta')^2-2\Delta'\sigma'+(\sigma')^2.\end{aligned} \tag{5.9.49}$$

Therefore, it suffices to show that

$$\alpha'\beta'+\Delta'\sigma'=\lambda'\rho'. \tag{5.9.50}$$

Indeed, we have

$$\lambda'\rho'=(x'y+z'w)(y'x+w'z)=x'xyy'+x'yw'z+z'wy'x+z'ww'z, \tag{5.9.51}$$

whereas

$$\begin{aligned}\alpha'\beta'+\Delta'\sigma' =&(xx'-ww')(yy'-zz')+(x'z+y'w)(z'x+w'y)\\ =&xx'yy'-xx'zz'-ww'yy'+ww'zz'+x'xzz'+\\ &+x'zw'y+y'wz'x+yy'ww'\\ =&x'xyy'+ww'zz'+x'zw'y+y'wz'x\\ =&\lambda'\rho'.\end{aligned} \tag{5.9.52}$$

Therefore,

$$\mathbf{K}^2=-\mathbf{J}^2. \tag{5.9.53}$$